

INTEGER VALUED BETTING STRATEGIES AND TURING DEGREES

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ABSTRACT. Betting strategies are often expressed formally as martingales. A martingale is called integer-valued if each bet must be an integer value. Integer-valued strategies correspond to the fact that in most betting situations, there is a minimum amount that a player can bet. According to a well known paradigm, algorithmic randomness can be founded on the notion of betting strategies. A real X is called integer-valued random if no effective integer-valued martingale succeeds on X . It turns out that this notion of randomness has interesting interactions with genericity and the computably enumerable degrees. We investigate the computational power of the integer-valued random reals in terms of standard notions from computability theory.

1. INTRODUCTION

An interesting strategy for someone who wishes to make a profit by betting on the outcomes of a series of unbiased coin tosses, is to double the the amount he bets each time he places a bet. Then, independently of whether he bets on heads or tails, if the coin is fair (i.e. the sequence of binary outcomes is random) he is guaranteed to win infinitely many bets. Furthermore, each time he wins he recovers all previous losses, plus he wins a profit equal to the original stake. This is a simple example from a class of betting strategies that originated from, and were popular in 18th century France. They are known as *martingales*. The “success” of this strategy is essentially equivalent to the fact that a symmetric one-dimensional random walk will eventually travel an arbitrarily long distance to the right of the starting point (as well as an arbitrarily long distance to the left of the starting point).

So what is the catch? For such a strategy to be maintained, the player needs to be able to withstand arbitrarily large losses, and such a requirement is not practically feasible. In terms of the random walk, this corresponds to the fact that, before it travels a large distance to the right of the starting point, it is likely to have travelled a considerable distance to the left of it.

1.1. Martingales and randomness. Martingales have been reincarnated in probability theory (largely though the work of Doob), as (memoryless) stochastic processes (Z_n) such that the conditional expectation of each Z_{n+1} given Z_n remains equal to the expectation of Z_0 . The above observations on a fair coin-tossing game are now theorems in the theory of martingales. For example, Doob’s maximal martingale inequality says that with probability 1, a non-negative martingale is bounded. Intuitively this means that, if someone is not able (or willing) to take credit (so that he continues to bet after his balance is negative) then the probability that he makes an arbitrarily large amount in profit is 0.

Martingales in probability rest on a concept of randomness in order to determine (e.g. with high probability) or explain the outcomes of stochastic processes. It turns out that this methodology can be turned upside down, so that certain processes are used in order to define or explain the concept of randomness. Such an approach was initiated by Schnorr in [Sch71], and turned out to be one of the standard and most intuitive methods of assigning meaning to the concept of randomness for an individual string or a real (i.e. an infinite binary sequence, a point in the Cantor space). This approach is often known as the *unpredictability paradigm*, and it says that it should not be possible for a computable predictor to be able to predict bit $n + 1$ of a real X based on knowledge of bits $1, \dots, n$ of

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X , namely $X \upharpoonright n$. The unpredictability paradigm can be formalized by using *martingales*, which (for our purposes) can be seen as betting strategies. We may define a martingale to be a function $f : 2^{<\omega} \rightarrow \mathbb{R}^{\geq 0}$ which obeys the following fairness condition:

$$f(\sigma) = \frac{f(\sigma 0) + f(\sigma 1)}{2}.$$

If f is partial, but its domain is downward closed with respect to the prefix relation on finite strings, then we say that f is a *partial martingale*.

In probability terms, f can be seen as a stochastic process (a series of dependent variables) Z_s where Z_s represents the capital of a player at the end of the s th bet (where there is 50% chance for head or tails). Then the fairness condition says that the expectation of f at stage $s + 1$ is the same as the value of f at stage s . In other words, the fairness condition says that the expected growth of f at each stage of this game is 0. If we interpret f as the capital of a player who bets on the outcomes of the coin tosses, the fairness condition says that there is no bias in this game toward the player or the house. Moreover note that our definition of a martingale as a betting strategy requires that it is non-negative. Recalling our previous discussion about gambling systems, this means that we do not allow the player to have a negative balance. This choice in the definition is essential, as it prevents the success of a ‘martingale betting system’ as we described it. Continuing with our definition of martingales as betting strategies, we say that f succeeds on a real X if

$$\limsup_{n \rightarrow \infty} f(X \upharpoonright n) = \infty.$$

Schnorr [Sch71] was interested in an algorithmic concept of randomness. Incidentally, Martin-Löf [ML66] had already provided a mathematical definition of randomness based on computability theory and effective measure theory. But Schnorr wanted to approach this challenge via the intuitive concept of betting strategies. He proved that a real (i.e. an infinite binary sequence) X is Martin-Löf random if and only if no effective martingale can succeed on it. Here “effective” means that f is computably approximable from below. Schnorr’s result is an effective version of the maximal inequality for martingales in probability theory, which says that with probability 1 a non-negative martingale is bounded. There is a huge literature about the relationship between martingales and effective randomness, and variations on the theme, such as computable martingales and randomness, partial computable martingales, nonmonotonic martingales, polynomial time martingales, etc. We refer the reader to Downey and Hirschfeldt [DH10] and Nies [Nie09] for some details and further background.

1.2. Why integer-valued martingales? Recall the standard criticism of martingale betting systems, i.e. that their success depends on the ability of the player to sustain arbitrarily large losses. This criticism lead (for the purpose of founding algorithmic randomness) to defining a martingale as a function from the space of coin-tosses to the non-negative reals (instead of all the reals) which represent the possible values of the capital available to the player. There is another criticism on such betting strategies that was not taken into account in the formal definition. Schnorr’s definition of a martingale (as a betting strategy) allows betting infinitesimal (i.e. arbitrarily small) amounts. Clearly such an option is not available in real gambling situations, say at a casino, where you cannot bet arbitrarily small amounts on some outcome. It becomes evident that restricting the betting strategies to a discrete range results in a more realistic concept of betting. Such considerations led Bienvenu, Stephan and Teutsch [BST12] to introduce and study *integer-valued martingales*, and the corresponding randomness notions. Interestingly, it turns out that the algorithmic randomness based on integer-valued martingales is quite different from the theory of randomness based on Martin-Löf [ML66] or Schnorr [Sch71] (as developed in the last 30 years, see [DH10, Nie09] for an overview). The reason for this difference is that most of the classical martingale arguments in algorithmic randomness make substantial use of the property of being able to bet infinitesimal amounts (thereby effectively avoiding bankruptcy at any finite stage of the process).

Quite aside from the motivations of examining the concept of integer-valued martingales for its own sake, if we are to examine the randomness that occurs in practice, then such discretised randomness will be the kind we would get. The reason is evident: we can only use a finite number of rationals for our bets, and these scale to give integer

Integer-valued	Computable
Finite-valued	Single-valued
Partial integer-valued	Partial computable

Table 1. Randomness notions based on effective martingales

values. *Additionally*, at the more speculative level, if the universe is granular, finite, and not a manifold, then if there is any randomness to be had (such as in quantum mechanics) it will be integer-valued for the same reason.

1.3. Integer randomness notions and computability. We formally introduce and discuss the notions of integer-valued randomness in the context of computability theory. For the purposes of narrative flow, we will assume that the reader is familiar with the basics of algorithmic randomness. Schnorr based algorithmic randomness on the concept of effective strategies. Along with this foundational work, he introduced and philosophically argued for a randomness notion which is weaker than Martin-Löf randomness and is now known as *Schnorr randomness*. Further notions, like *computable randomness*, are quite natural from the point of view of betting strategies and have been investigated extensively. Integer-valued martingales induce randomness notions with properties that are quite a different in flavour from those of, for instance, Martin-Löf randomness, computable randomness and the like. Our goal in the present paper is to clarify the relationship between integer-valued randomness and classical degree classes which measure levels of computational power.

Definition 1.1 (Integer-valued martingales). *Given a finite set $F \subseteq \mathbb{N}$, we say that a martingale f is F -valued if $f(\sigma i) = f(\sigma) \pm k$ for some $k \in F$. A martingale is integer-valued if it is \mathbb{N} -valued, and is single-valued if $F = \{a, 0\}$ for some $a \neq 0$.*

Note that a martingale is F -valued if at any stage we can only bet k dollars for some $k \in F$ on one of the outcomes $i \in \{0, 1\}$, and must lose k dollars if $1 - i$ is the next bit. We note that partial integer-valued martingales are defined as in Definition 1.1, only that the martingales can be partial. In the following we often say that, given a string σ , the string $\sigma 0$ is the *sibling* of $\sigma 1$ (and $\sigma 1$ is the *sibling* of $\sigma 0$).

If we restrict our attention to the countable class of computable or partial computable martingales, we obtain a number of algorithmic randomness notions. For example, a real is [partial] computably random if no [partial] computable martingale succeeds on it. Similar notions are obtained if we consider integer-valued martingales.

Definition 1.2 (Integer-valued randomness). *A real X is [partial] integer-valued random if no [partial] computable integer-valued martingale succeeds on it. Moreover X is finitely-valued random if for each finite set $F \subseteq \mathbb{N}$, no computable F -valued martingale succeeds on it, and is single-valued random if no computable single-valued martingale succeeds on it.*

We list these randomness notions in Table 1, along with the traditional randomness notions computable and partial computable randomness. Note that partial integer-valued randomness is stronger than integer-valued randomness, just as partial computable randomness is stronger than computable randomness. Bienvenu, Stephan and Teutsch [BST12] clarified the relationship between integer-valued, single-valued and a number of other natural randomness notions. Figure 1 illustrates some of the implications that they obtained. We already know (see [DH10]) that computable randomness implies Schnorr randomness, which in turn implies Kurtz randomness (with no reversals) and that Schnorr randomness implies the law of large numbers. Bienvenu, Stephan and Teutsch proved that if we add the above notions to the diagram in Figure 1, no other implications hold apart from the ones in the diagram and the ones just mentioned. In addition, we may add a node for ‘partial computably random’ and an arrow from it leading to the node ‘computably random’. Nies showed in [Nie09, Theorem 7.5.7] that the converse implication does not hold, i.e. there are computably random reals which are not partial computably random. A strong version

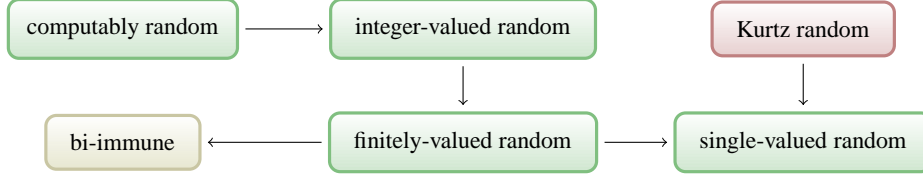


Figure 1. Implications between randomness notions obtained in [BST12].

of this fact holds for integer-valued randomness. We show that there are integer-valued random reals which not only are not partial integer-valued random, but they do not contain any partial integer-valued random reals in their Turing degree.

One interesting observation of Bienvenu, Stephan and Teutsch [BST12], was that integer-valued was a meeting point of genericity (and hence category) and measure since weakly 2-generic sets are integer-valued random. Hence the integer-valued randoms are co-meager as well as having measure 1. The reader should recall that a subset of \mathbb{N} is called n -generic if it meets or avoids all Σ_n^0 sets of strings, and is weakly n -generic if it meets all dense Σ_n^0 sets of strings. The reason that highly generic reals are integer-valued random is that there is a finitary strategy to make a set integer-valued random, since we can force an opponent who bets to lose. The point here is that if the minimum bet is one dollar and he has k dollars to spend, then he can lose at most k times, so we can use a finite strategy to force the opponent into a cone (in Cantor space) where he cannot win. This finite strategy is not available if arbitrarily small bets are allowed. Naturally the question arises as to what level of genericity is needed for constructing integer-valued randoms. Bienvenu, Stephan and Teutsch [BST12] proved that it is possible to have a 1-generic which was *not* integer-valued random. So the answer they gave to this question is ‘somewhere between weak 2 and 1-genericity’. As we see in the next section, we give a more precise answer.

1.4. Our results, in context. Bienvenu, Stephan and Teutsch [BST12] showed that the class of integer-valued random is co-meager, so sufficient genericity is a guarantee for this kind of randomness. They also quantified this statement via the hierarchy of genericity, showing that the genericity required lies somewhere between weak 2-genericity and 1-genericity. We show that a notion of genericity from [DJS96] which is known as *pb*-genericity, implies (partial) integer-valued randomness. Recall from [DJS96] that set of strings S is *pb*-dense if it contains the range of a function $f : 2^{<\omega} \rightarrow 2^{<\omega}$ with a computable approximation (f_s) such that $f(\sigma) \supseteq \sigma$ for all strings σ and $|\{s \mid f_{s+1}(\sigma) \neq f_s(\sigma)\}| \leq h(\sigma)$ where the function $h : 2^{<\omega} \rightarrow \omega$ is primitive recursive. A real X is *pb*-generic if every *pb*-dense set of strings contains a prefix of X .

Theorem 1.3 (Genericity for integer-valued randoms). *Every pb-generic real is (partial) integer-valued random.*

This result might suggest that integer-valued randomness and partial integer-valued randomness are not easily distinguishable. In Section 2.2 we present a rather elaborate finite injury construction of a real which is integer-valued random but not partial integer-valued random. In Section 2.3 this construction is modified into a $0''$ tree argument, which proves the following degree separation between the two randomness notions.

Theorem 1.4 (Degree separation of randomness notions). *There exists an integer-valued random $X <_T 0'$ which does not compute any partial integer-valued random.*

We are interested in classifying the computational power that is associated with integer-valued randomness. Computational power is often represented by properties of degrees, which in turn define classes like the degrees which can solve the halting problem, or the array non-computable degrees from [DJS96]. The reader might recall that A is array noncomputable if and only if for all $f \leq_{wt} 0'$, there is a function $g \leq_T A$ such that $g(n) > f(n)$ for infinitely many n . This class has turned out to be ubiquitous and characterized classes defined by many distinct combinatorial properties. Recall that a presentation of a real A is a c.e. prefix-free set of strings representing an open set of Lebesgue measure α . The c.e. array noncomputable degrees are exactly the c.e. degrees that

- (a) contain c.e. sets A of infinitely often maximal (i.e. $2 \log n$) Kolmogorov complexity; (Kummer [Kum96])
- (b) have effective packing dimension 1; (Downey and Greenberg [DG08])
- (c) compute left-c.e. reals $B <_T A$ such that every presentation of A is computable from B ; (Downey and Greenberg [DN14])
- (d) compute a pair of disjoint c.e. sets such that every separating set for this pair computes the halting problem; (Downey, Jockusch, and Stob [DJS90])
- (e) do not have strong minimal covers; (Ishmukhametov [Ish99])

Also by Cholak, Coles, Downey, Herrmann [CCDH01] the array noncomputable c.e. degrees form an invariant class for the lattice of Π_1^0 classes via the thin perfect classes.

Theorem 1.3 can be used to show that a large class of degrees compute (partial) integer-valued randoms. By [DJS96], every array noncomputable degree computes a pb -generic. Therefore Theorem 1.3 has the following consequence.

Corollary 1.5 (Computing integer-valued randoms). *Every array noncomputable degree computes a (partial) integer-valued random.*

Note however that an integer-valued random need not be of array noncomputable degree. Indeed, it is well known that there are array computable Martin-Löf randoms. A converse of Corollary 1.5 can be obtained for the c.e. degrees, as Theorem 1.7 shows.

While genericity is an effective tool for exhibiting integer-valued randomness in the global structure of the degrees (as we demonstrated above) it is incompatible with computable enumerability. Since generic degrees (even 1-generic) are not c.e., investigating integer-valued randomness in the c.e. degrees requires a different analysis. Already the fact that randomness can be exhibited in the c.e. degrees is quite a remarkable phenomenon, and restricted to weak versions of randomness. Martin-Löf randomness is the strongest standard randomness notion that can be found in the c.e. degrees. A c.e. degree contains a Martin-Löf random set if and only if it is complete (i.e. it is the degree of the halting problem). Furthermore, the complete c.e. degree contains the most well-known random sequence—Chaitin’s Ω —which is the measure of the domain of a universal prefix-free machine, the universal *halting probability*. An interesting characteristic of this random number is that it is left-c.e., i.e. it can be approximated by a computable increasing sequence of rationals. Weaker forms of randomness—like computable and Schnorr randomness—can be found in incomplete c.e. degrees, even in the form of left-c.e. sets. For example, Nies, Stephan and Terwijn [NST05] showed that the c.e. degrees which contain computably and Schnorr random sets are exactly the high c.e. degrees. Moreover each high c.e. degree contains a computably random left-c.e. set and a Schnorr random left-c.e. set. We prove an analogous result for integer-valued randomness and partial integer-valued randomness.

Theorem 1.6 (C.e. degrees containing integer-valued random left-c.e. sets). *A c.e. degree contains a (partial) left-c.e. integer-valued random if and only if it is high.*

This result is pleasing, but this is where the similarities between computable randomness (based on computable betting strategies) and integer-valued randomness (based on integer-valued computable betting strategies) end, at least with respect to the c.e. degrees. We note that

$$(1.1) \quad \left\{ \begin{array}{l} \text{In the c.e. degrees, the following classes are equal to the high degrees:} \\ \quad (i) \text{ degrees containing computably random sets;} \\ \quad (ii) \text{ degrees containing left-c.e. computably random sets;} \\ \quad (iii) \text{ degrees computing computably random sets.} \end{array} \right.$$

This characterization follows from the following facts, where (a) is by Schnorr [Sch71], (c) was first observed by Kučera [Kuč85], and (b), (d) are from [NST05].

- (a) computable randomness implies Schnorr randomness;
- (b) a Schnorr random which does not have high degree is Martin-Löf random;
- (c) a Martin-Löf random of c.e. degree is complete;

(d) every high c.e. degree contains a computably random left-c.e. set.

In the case of integer-valued randomness (1.1) fails significantly. In particular, the c.e. degrees that compute integer-valued randoms are not the same as the c.e. degrees that contain integer-valued randoms. In fact, we provide the following characterization of the c.e. degrees that compute integer-valued randoms.

Theorem 1.7 (C.e. degrees computing integer-valued randoms). *A c.e. degree computes an integer-valued random if and only if it is array noncomputable.*

In view of this result it is tempting to think that every c.e. array noncomputable might contain an integer-valued random. We will see however in Section 5.1 that there are array noncomputable c.e. degrees which do not contain integer-valued randoms. In fact, Theorem 1.8 is an extreme version of this fact, which is tight with respect to the jump hierarchy.

We have seen that high c.e. degrees are powerful enough to contain integer-valued randoms, even left-c.e. integer-valued randoms. However, while we know that left-c.e. integer-valued randoms necessarily have high degree, the question arises as to whether a weaker jump class is sufficient to guarantee that a c.e. degree contains an integer-valued random, if we no longer require that the set is left-c.e. We give a negative answer to this question by the following result, which we prove in Section 5.2 by a $0'''$ -argument.

Theorem 1.8. *There is a high₂ c.e. degree which does not contain any integer-valued randoms.*

Note that this result shows the existence of array noncomputable c.e. degrees which do not contain integer-valued randoms, in stark contrast to Theorem 1.7.

Furthermore, the c.e. degrees that contain integer-valued randoms are not the same as the c.e. degrees that contain left-c.e. integer-valued randoms. In fact, in stark contrast with Theorem 1.6, there exists a low c.e. degree containing an integer-valued random set. More generally, we can find c.e. degrees containing integer-valued random sets in every jump class. Section 4 is devoted to the proof of this result.

Theorem 1.9 (Jump inversion for c.e. integer-valued random degrees). *If \mathbf{c} is c.e. in and above $0'$ then there is an integer-valued random A of c.e. degree with $A' \in \mathbf{c}$.*

There are a number of open questions and research directions pointed by the work in this paper. For example, is there a c.e. degree which contains an integer-valued random but does not contain any partial integer-valued randoms? More generally, which degrees contain partial computable randoms? Algorithmic randomness based on partial martingales is a notion that remains to be explored on a deeper level.

2. GENERICITY AND PARTIAL INTEGER-VALUED RANDOMS

Bienvenu, Stephan, Teutsch [BST12, Theorem 8] showed that every weakly 2-generic set is integer-valued random. In Section 2.1 we give a proof of Theorem 1.3, i.e. that pb -genericity is sufficient for (partial) integer-valued randomness.

Hence a certain notion of genericity (pb -genericity) is a source of integer-valued randomness. In fact, by Theorem 1.3, every pb -generic is not only integer-valued random but also partial integer-valued random. We do not have concrete examples of reals that are integer-valued random but not partial integer-valued random. Section 2.2 is dedicated to constructing such an example. We give the basic construction of a Δ_2^0 real which is integer-valued random but not partial integer-valued random. We give this in full detail, as it is based on an interesting idea.

In Section 2.3 we provide the necessary modification of the previous construction in order to show that the degrees of integer-valued randoms and partial integer-valued randoms can also be separated, even inside Δ_2^0 . In particular, we are going to prove Theorem 1.4, i.e. that there is a Δ_2^0 integer-valued random which does not compute any partial integer-valued randoms. These modifications are essentially the implementation of the main argument of Section 2.2 on a tree, which results from the additional requirements that introduce infinitary outcomes that need to be guessed (i.e. the totality of the various functionals with oracle the constructed set). Given the original strategies and construction, the tree argument is fairly standard.

2.1. Proof of Theorem 1.3. For every partial computable integer-valued martingale m with effective approximation (m_s) we define a function $\hat{m} : 2^{<\omega} \rightarrow 2^{<\omega}$ with uniformly computable approximation (\hat{m}_s) . Let $\hat{m}_0(\sigma) = \sigma$ for all strings σ . Inductively in s , suppose we have defined \hat{m}_s . At stage $s + 1$, if $m_{s+1}(\hat{m}_s(\sigma))$ is defined and there exists an extension τ of it of length $\leq s + 1$ such that $m_{s+1}(\tau)$ is defined and $m_{s+1}(\tau) < m_{s+1}(\hat{m}_s(\sigma))$, then we define $\hat{m}_{s+1}(\sigma)$ to be the least such string τ (where strings are ordered first by length and then lexicographically). Otherwise let $\hat{m}_{s+1}(\sigma) = \hat{m}_s(\sigma)$.

Since m is an integer-valued martingale we have $|\{s \mid \hat{m}_{s+1}(\sigma) \neq \hat{m}_s(\sigma)\}| \leq m(\sigma) + 1$ and $m(\sigma) \leq 2^{|\sigma|} \cdot m(\emptyset)$. Note that given any partial computable martingale m , the range of \hat{m} is dense. So every pb -generic intersects the range of \hat{m} for every partial computable martingale m . Moreover given any partial computable martingale m , the range of \hat{m} is a subset of

$$W_m = \{\sigma \mid m(\sigma') \simeq m(\sigma) \text{ for all extensions } \sigma' \text{ of } \sigma\}.$$

So every pb -generic intersects W_m for each partial computable martingale m . This means that every pb -generic is partial integer-valued random.

2.2. An integer-valued random which is not partial integer-valued random. It suffices to construct an integer-valued random set $A \leq_T \emptyset'$ and a partial integer-valued martingale m which succeeds on A . We will define a computable approximation (A_s) which converges to a set A which has the required properties. Let $\langle n_e \rangle_{e \in \omega}$ be an effective list of all partial computable integer-valued martingales. For each $e \geq 2$ we need to satisfy the requirements

R_e : If n_e is total, then n_e does not succeed on A .

Q_e : m wins (at least) $\$e$ on A .

We first see how we might meet one requirement R_0 . We begin by setting $A_0 = 1^\omega$ and defining m to start with $\$2$ and wager $\$1$ on every initial segment of A_0 . If we later see that n_0 has increased its capital along A_0 , then we would like to move our approximation to A so that n_0 decreases in capital. In changing A_s to decrease n_0 's capital we may also decrease m 's capital along A_s . If n_e had, say, $\$10$ in capital at some point, then as it loses at least a dollar every time it decreases in capital, it can lose at most 10 times. Our martingale, if it bets in $\$1$ wagers, can withstand losing $\$10$ only if its capital at that point is at least $\$11$.

If m does not have sufficient capital for us to start attacking immediately, we must find a way to increase its capital. We can increase m 's capital to $\$k$ as follows. We have not yet defined m on any string extending 0. We therefore wait until n_0 has halted on all strings of length k . If this never happens, then n_0 is not total, and R_0 is met. If n_0 does halt on all strings of length k , we pick a string τ of length k extending 0 such that $n_0(\tau) \leq n_0(0)$. Such a string must exist since n_0 is a (partial) martingale. We are then free to define m to wager $\$1$ on every initial segment of τ and set $A_s = \tau \hat{1}^\omega$. We will then have $m(\tau) > n_0(\tau)$. If n_0 later increases its capital along A_s we will be able to change the approximation to A to decrease n_0 's capital. As m now has greater capital than n_0 , we will be able to decrease n_0 's capital to $\$0$ while ensuring that m does not run out of money.

When dealing with multiple requirements, we must take care in defining m as it is a global object. We set a restraint r_e for every $e \in \omega$. We arrange things so that only R_e will be able to define m on a string extending $A_s \upharpoonright r_{e,s} \hat{0}$. Suppose R_e has not required attention since it was last injured. When we see n_e increase its capital above $n_e(A_s \upharpoonright r_e)$, rather than starting to attack immediately, even if possible, we choose a string τ' extending $A_s \upharpoonright r_{e,s} \hat{0}$ and define m such that $m(\tau') - m(A_s \upharpoonright r_{e,s} \hat{0}) > n_e(A_s \upharpoonright r_{e,s} \hat{0})$. We injure requirements of weaker priority by lifting their restraints. We may then decrease n_e 's capital to $\$0$ and still have m left with some capital. We now turn to the formal details of the construction.

We have for every requirement R_e a restraint r_e . At every stage a requirement will either be declared to be waiting for convergence at some length, or declared to not be waiting for convergence at any length. As usual, this will stay in effect at the next stage unless otherwise mentioned. We say that R_e requires attention at stage s if either

- (i) R_e was declared to not be waiting for convergence at any length at stage s , and there is l such that
 - (a) $l > r_{e,s}$,
 - (b) $n_{e,s}(\sigma) \downarrow$ for all strings σ of length l , and

- (c) $n_e(A_s \upharpoonright l) > n_e(A_s \upharpoonright (l-1))$, or
- (ii) R_e was declared to be waiting for convergence at length h at stage s , and $n_{e,s}(\sigma) \downarrow$ for all strings σ of length h .

In Case (i) we say that R_e requires attention through l . We say that Q_e requires attention at stage s if $m(A_s \upharpoonright r_{e,s}) < e$. We order the requirements as $R_0, R_1, R_2, Q_2, R_3, Q_3, \dots$.

Construction. *Stage 0:* Set $A_0 = 1^\omega$ and $m(\lambda) = 2$, and let m wager \$1 on every initial segment of A_0 . Set $r_{e,0} = e$ for all $e \in \omega$. For all $e \in \omega$, declare that R_e is not waiting for convergence at any length at stage 1.

Stage s , $s \geq 1$: Find the requirement of strongest priority which requires attention at stage s . (If no such requirement exists, go to the next stage.) There are several cases.

Case 1: R_e requires attention at stage s in Case (i). Has R_e required attention since it was last injured?

Subcase 1a: No. Declare R_e to be waiting for convergence at length $n_e((A_s \upharpoonright r_{e,s})^\wedge 0) + 1 + (r_{e,s} + 1)$ at stage $s + 1$.

Subcase 1b: Yes. Suppose l is least such that R_e requires attention through l at stage s . Let $A_{s+1} = (A_s \upharpoonright (l-1))^\wedge (1 - A_s(l-1))^\wedge 1^\omega$. Define m to wager \$1 on every initial segment of A_{s+1} of length at least $l + 1$. For all $e' > e$, let $r_{e',s+1}$ be a fresh large number such that for all $e_1 < e_2$ we have $r_{e_1,s+1} < r_{e_2,s+1}$, and declare that $R_{e'}$ is not waiting for convergence at any length at stage $s + 1$.

Case 2: R_e requires attention at stage s in Case (ii). Suppose R_e was declared to be waiting for convergence at length h at stage t .

Choose a string τ above $A_s \upharpoonright r_{e,s}^\wedge 0$ of length h such that $n_e(\tau) \leq n_e(A_s \upharpoonright r_{e,s}^\wedge 0)$. Define m to wager \$1 along every initial segment of τ with length in $(r_{e,s} + 1, |\tau|]$. Set $A_{s+1} = \tau^\wedge 1^\omega$. Define m to wager \$1 on every initial segment of A_{s+1} of length at least $|\tau| + 1$. For all $e' > e$, let $r_{e',s+1}$ be a fresh large number such that if $e_1 < e_2$ then $r_{e_1,s+1} < r_{e_2,s+1}$. For all $e'' \geq e$, declare that $R_{e''}$ is not waiting for convergence at any length at stage $s + 1$.

Case 3: Q_e requires attention at stage s . Let τ be the least string extending $A_s \upharpoonright r_{e,s}$ for which $m(\tau) = e$. Set $A_{s+1} = \tau^\wedge 1^\omega$ and for all $e' > e$, let $r_{e',s+1}$ be a fresh large number such that if $e_1 < e_2$ then $r_{e_1,s+1} < r_{e_2,s+1}$. For all $e'' > e$, declare that $R_{e''}$ is not waiting for convergence at any length at stage $s + 1$.

Verification. Before we demonstrate the satisfaction of requirements R_e, Q_e , we need to show that the partial martingale m is well-defined, and for all $n, s \in \omega$,

$$(2.1) \quad \text{if } m(A_s \upharpoonright n) \downarrow \text{ then } m(A_s \upharpoonright n) \geq 1.$$

We clarify that in this statement, m denotes the state of the partial martingale at stage s . We prove this statement by induction on the stages s . We first claim that if R_e requires attention in Case (ii) at stage s , then m has not been defined on any string extending $A_s \upharpoonright r_{e,s}^\wedge 0$. Suppose R_e requires attention in Case (ii) at stage s . Let $s^* - 1 < s$ be the last stage at which R_e was initialised. We choose r_{e,s^*} to be some fresh large number. In particular, m has not been defined on any string extending $A_{s^*} \upharpoonright r_{e,s^*}^\wedge 0$. Note that $r_{e,s} = r_{e,s^*}$ and $A_{s^*} \upharpoonright r_{e,s} = A_s \upharpoonright r_{e,s}$. If R_k for $k > e$ acts at stage $t > s^*$ it may define m on $A_t \upharpoonright r_{k,t}^\wedge 0$, but as $r_{k,t} > r_{e,t} \geq r_{e,s}$, it cannot define m on any string extending $A_{s^*} \upharpoonright r_{e,s}^\wedge 0$. No R_i for $i < e$ may act between stages s^* and s as this contradicts the choice of s^* . Therefore m has not been defined on any string extending $A_s \upharpoonright r_{e,s}^\wedge 0$ at stage s .

By the definition of A_0 and m at stage 0, we have $m(A_0 \upharpoonright n) \geq 2$ for all n . Furthermore, m is at least 1 on the sibling of any initial segment of A_0 (recall the definition of the *sibling* of a string, just after Definition 1.1). Suppose that $m(A_s \upharpoonright n) \geq 1$ for all n and m is at least 1 for any sibling of an initial segment of A_s . If we act for Q_e at stage s , then it is easy to see that $m(A_{s+1} \upharpoonright n) \geq 1$ for all n . So suppose that we act for R_e at stage s . If we act in Case 2 for R_e at stage s , then we will choose a string τ extending $A_s \upharpoonright r_{e,s}^\wedge 0$ of some length h . By assumption, $m(A_s \upharpoonright r_{e,s}^\wedge 0) \geq 1$. We then let A_{s+1} extend τ and define m such that $m(\tau) - m(A_s \upharpoonright r_{e,s}^\wedge 0) > n_e(A_s \upharpoonright r_{e,s}^\wedge 0)$. Then $m(A_{s+1} \upharpoonright n) \geq 1$ for all n .

Now suppose that we act for R_e in Subcase 1b through length l at stage s . Our martingale m wagers at most \$1 at a time, and so loses at most \$1 at a time. We decrease n_e by at least \$1 while decreasing m by at most \$1. As $m(A_s \upharpoonright l) > n_e(A_s \upharpoonright (l-1))$, we may reduce n_e 's capital to \$0 while m has capital remaining. Now requirements of

stronger priority than R_e may start to act. Suppose that $R_{e'}$ with $e' < e$ requires attention. If $R_{e'}$ requires attention in Case (ii) then we will act as in the previous paragraph and m will still have capital left. Otherwise, $R_{e'}$ may act in Subcase 1b at stage t after having acted in Case 2 *before* stage s . However, in this case, we would have increased m 's capital by $n_{e'}(A_s \upharpoonright r_{e',s} \hat{\ } 0)$ previously. Therefore, after having reduced n_e 's capital to 0, we may then reduce $n_{e'}$'s capital to 0 as well, while ensuring that m still has capital remaining. This concludes the induction on the stages and the proof of (2.1).

Note that we have not yet shown that the approximation (A_s) converges to a set A . This is a consequence of the use of restraints in the construction, and the following lemma which says that each requirement R_e receives attention only finitely often.

Lemma 2.1. *For all $e \in \omega$, R_e receives attention only finitely often, and is met.*

Proof. Suppose by induction that $s^* - 1$ is the last stage at which R_e is injured (i.e. the least stage after which no requirement of stronger priority than R_e receives attention). If R_e never requires attention at some later stage in Case (i), then either n_e is not total, or $n_e(A) \leq n_e(A_{s^*} \upharpoonright r_{e,s^*})$. In either case R_e is met. Therefore suppose that R_e requires attention through l at some stage $s' \geq s^*$. We will act in Subcase 1a and declare R_e to be waiting for convergence at length $n_e((A_{s'} \upharpoonright r_{e,s'}) \hat{\ } 0) + 1 + (r_{e,s'} + 1) =: h_0$. As no requirement $R_{e'}$ for $e' < e$ receives attention after stage s^* , R_e will be waiting for convergence at length h_0 until, if ever, R_e requires attention in Case (ii). If R_e never requires attention after stage s' then n_e is not total. So suppose that R_e requires attention in Case (ii) at stage t' . We choose a string τ of length h_0 above $A_{t'} \upharpoonright r_e \hat{\ } 0$ such that $n_e(\tau) \leq n_e(A_{t'} \upharpoonright r_e \hat{\ } 0)$. Since n_e is a (partial) martingale, $n_e(\sigma) \leq n_e(A_{t'} \upharpoonright r_e \hat{\ } 0)$ for at least one string σ of length h_0 above $A_{t'} \upharpoonright r_e \hat{\ } 0$. Therefore such a string must exist.

We set $A_{t'+1} = \tau \hat{\ } 1^\omega$ and define m so that $m(\tau) > n_e(\tau)$. If R_e receives attention after stage t' then it must do so in Subcase 1b. Our martingale m wagers at most \$1 at a time, and so loses at most \$1 at a time. If R_e requires attention through some $l > h_0$ at a stage $t'' > t'$ then we will again act in Subcase 1b and force n_e to lose at least \$1 while m loses at most \$1. This can happen at most $n_e(\tau)$ many times before n_e loses all its capital and can no longer bet. Thus the induction can continue, and R_e is met. \square

It remains to show that m succeeds on A . For this, it suffices to show that all requirements Q_e , $e \geq 2$ are met.

Lemma 2.2. *For all $e \geq 2$, Q_e receives attention only finitely often, and is met.*

Proof. Suppose by induction that s^* is the least stage after which no requirement of stronger priority than Q_e receives attention. As no requirement of stronger priority than Q_e receives attention after stage s^* , the restraint r_{e,s^*} will never again be increased unless Q_e acts. If Q_e requires attention at stage $s > s^*$ then we must have $m(A_s \upharpoonright r_{e,s}) < e$. As no requirement of stronger priority receives attention after stage s^* , we must have that Q_{e-1} is satisfied, and so $m(A_s \upharpoonright r_{e,s}) = e - 1$. We have defined m to wager \$1 on all initial segments of A_s and so there is τ such that $m(\tau) = e$. We let $A_{s+1} = \tau \hat{\ } 1^\omega$ and increase the restraints $r_{e',s+1}$ for all $e' \geq e$. We then have that $\tau < A$ and Q_e is satisfied. \square

2.3. Integer-valued randoms not computing partial integer-valued randoms. The construction of Section 2.2, non-trivial as it is, admits some modifications. For example, it is not hard to add the requirement that A is 1-generic and still successfully perform the argument. This requirement can be canonically split into an infinite sequence of conditions, with corresponding strategies in the constructions which will occasionally change the approximation to A . Since the 1-genericity sub-requirements are finitary, their effect will be similar to (in fact, more benign than) the R_e requirements.

There is a 1-generic integer-valued random which is not partial integer-valued random.

Note that 1-genericity is generalized low, so since A is Δ_2^0 , it follows that

There is a low integer-valued random which is not partial integer-valued random.

The next modification of the construction of Section 2.2 results in the proof of Theorem 1.4 and requires more explanation. We can replace the requirements Q_e with

$$Q_{e,k}^*: \text{ If } \Phi_e^A \text{ is total and non-computable then } m \text{ wins (at least) } \$k \text{ on } \Phi_e^A.$$

Note that now m will bet on Φ_e^A rather than A . For this reason, the family of requirements $Q_{e,k}^*$ need to act under the hypothesis that Φ_e^A is total. This means that we need to implement the argument of Section 2.2 on a tree, where the family of requirements $Q_{e,k}^*$ lies below a ‘mother-node’ Q_e^* which has two outcomes, a Π_2^0 outcome i and a Σ_2^0 outcome f . The outcome i corresponds to the fact that Φ_e^A has infinitely many expansionary stages (i.e. stages where the least n such that $\Phi_e^A(n)$ is undefined is larger than every before) while outcome f corresponds to the negation of this statement. Moreover the construction guarantees that if i is a true outcome, then Φ_e^A is total. Requirements $Q_{e,k}^*$ act as the Q_k of Section 2.2 while R_e are the same in the two constructions. Moreover these two requirements have a single outcome in the tree argument. The crucial point here is that if Φ_e^A is total and non-computable then Φ_e will have splitting along A , i.e. for each prefix τ of A there will be two finite extensions τ_i of τ and an argument x such that $\Phi_e^{\tau_i}(x) \neq \Phi_e^{\tau_1}(x)$. This means that before strategy $Q_{e,k}^*$ starts operating, it can secure a splitting which it can use to move away from versions of Φ_e^A on which m has not bet appropriately. In the construction of Section 2.2 this happened automatically as m bet on the real itself, and not its image under a Turing functional. Other than these points, the construction and verification is entirely similar to that of Section 2.2. Since this there is no novelty in this extension of the argument of Section 2.2 (given the standard machinery for tree arguments and the above remarks) we leave the remaining details to the motivated reader.

3. COMPUTABLE ENUMERABILITY AND INTEGER-VALUED RANDOMNESS

Nies, Stephan and Terwijn [NST05] showed that a c.e. degree is high if and only if it contains a computably random c.e. real. Moreover an analogous statement holds for partial computable random c.e. reals. In Section 3.1 we show that the same is true for integer-valued random c.e. reals. In other words, we give the proof of the first part of Theorem 1.6 that we discussed in the introduction. The proof of the remaining part of Theorem 1.6 (regarding partial integer-valued randomness) is deferred to Section 4.3, since the required machinery is similar to the one we use for the jump inversion theorems. In Section 3.2 we give the proof of Theorem 1.7. Note that by Corollary 1.5, for this proof it suffices to show that array computable c.e. degrees are not integer-valued random.

3.1. Degrees of left-c.e. integer-valued randomness. In this section we prove the first part of Theorem 1.6, i.e. that the high c.e. degrees are exactly those c.e. degrees which contain integer-valued random left-c.e. reals. The ‘if’ direction is a consequence of [NST05]. For the ‘only if’ direction it suffices to show that every integer-valued random left-c.e. real has high degree. Let α be an integer-valued random left-c.e. real, and $\langle \alpha_s \rangle_{s < \omega}$ a computable increasing sequence of rationals converging to α . We may assume that α has infinitely many 1s, and so by speeding up the enumeration we may ensure that α_s has at least s 1’s. Let $\text{Tot} = \{e \mid \varphi_e \text{ is total}\}$ be the canonical Π_2^0 complete set. We build a Turing functional Γ such that for all e , $\lim_k \Gamma^\alpha(e, k) = \text{Tot}(e)$. Then $\emptyset'' \leq_T \alpha'$ and so α is high. We also construct for each $e \in \omega$ a computable integer-valued martingale M_e . Let

$$\begin{aligned} d_{e,s} &= d_e[s] = \max\{k \mid (\forall \sigma \in 2^k)(M_e(\sigma)[s] \downarrow)\} \\ l_{e,s} &= \max\{k \mid (\forall j < k)(\varphi_e(j)[s] \downarrow)\}. \end{aligned}$$

We proceed in stages s , each consisting of two steps.

Construction at stage $s + 1$. For each $\langle e, k \rangle \leq s$ do the following

- If $\Gamma^\alpha(e, k)[s] \uparrow$, define it as follows. If $l_{e,s+1} \geq k$ let $\Gamma^\alpha(e, k)[s+1] = 1$ with use $\gamma(e, k)[s+1] = 0$; otherwise let $\Gamma^\alpha(e, k)[s+1] = 0$ with use the maximum of $\gamma(e, k)[s]$, $\gamma(e, k-1)[s+1]$, and h , where h is the position of the first 1 of α_s after $\alpha_s \upharpoonright d_{e,s}$.
- If $l_{e,s+1} > l_{e,s}$, Define M_e to wager 1 dollar on $(\alpha_s \upharpoonright h)^{\wedge} 1$, and bet neutrally on all other strings with length in $(d_{e,s}, h+1]$, where h is the position of the first 1 of α_s after $\alpha_s \upharpoonright d_{e,s}$.

Verification. Since α is a left-c.e. real, it follows from the construction that Γ is well defined, i.e. it is consistent. We show that Γ^α is total by showing that $\lim_s \gamma(e, k)[s]$ exists for all pairs (e, k) , and that $\lim_k \Gamma^\alpha(e, k) = \text{Tot}(e)$. We say that a stage t is *e-expansionary* if $l_{e,t} > l_{e,t-1}$.

First suppose that $e \notin \text{Tot}$. Then $\lim_s l_{e,s}$ and $\lim_s d_{e,s}$ both exist. Let $\lim_s l_{e,s} = l$ and $\lim_s d_{e,s} = d$, and suppose these limits are reached by stage s_0 . Let $s_1 \geq s_0$ be the least stage where $\alpha_{s_1} \upharpoonright d = \alpha \upharpoonright d$. Then for all k and all stages s where $\langle e, k \rangle \geq s \geq s_1$, $\Gamma^\alpha(e, k)[s]$ is set to 0 and $\gamma(e, k)[s]$ is set to be the position of the first 1 after $\alpha_s \upharpoonright d$. As α is left-c.e., the position of the first 1 of α_s after $\alpha_s \upharpoonright d$ at any stage $s \geq s_1$ is at most the position of the first 1 of α_{s_1} after $\alpha_{s_1} \upharpoonright d$. Therefore $\lim_s \gamma(e, k)[s]$ exists. For all k such that $\langle e, k \rangle < s_1$, $\lim_s \gamma(e, k)[s]$ is at most $\max_{s < s_1} h_{e,s}$ where $h_{e,s}$ is the position of the first 1 of α_s after $\alpha_s \upharpoonright d_{e,s}$.

Now suppose that $e \in \text{Tot}$. Then there is a sequence of stages $\langle s_i \rangle$ and a sequence $\langle h_i \rangle$ such that we define M_e to wager 1 dollar on $(\alpha_{s_i} \upharpoonright h_i)^\wedge 1$ at stage s_i . Note that s_i is least such that $l_{e,s_i} = i$. The real α is left-c.e., so $\alpha_s \upharpoonright (h_i + 1)$ can only move lexicographically to the left as s increases. Moreover, the approximation to α will never extend $(\alpha_{s_i} \upharpoonright h_i)^\wedge 0$, and so M_e cannot lose capital along α . As α is integer-valued random, M_e does not succeed on α . If $\alpha \upharpoonright (h_i + 1) = \alpha_{s_i} \upharpoonright (h_i + 1)$ then M_e increases in capital by 1 dollar. Therefore there are only finitely many h_i for which $\alpha \upharpoonright (h_i + 1) = \alpha_{s_i} \upharpoonright (h_i + 1)$. Let i_0 be least such that $\alpha \upharpoonright (h_j + 1) \neq \alpha_{s_j} \upharpoonright (h_j + 1)$ for all $j \geq i_0$. We show that $\Gamma^\alpha(e, k) = 1$ for all $k \geq i_0$, thus concluding the proof.

Suppose by induction that $\Gamma^\alpha(e, i) = 1$ for all $i_0 \leq i < k$. At stage s_{k-1} we have $l_{e,s_{k-1}} = k - 1$. For any stage t with $s_{k-1} \leq t < s_k$, if $\Gamma^\alpha(e, k)$ becomes undefined we set $\Gamma^\alpha(e, k)[t] = 0$ and set $\gamma(e, k)[t]$ to be the maximum of $\gamma(e, k)[t - 1]$, $\gamma(e, k - 1)[t]$, and the position of the first 1 of α_t after $\alpha_t \upharpoonright d_{e,t}$. Let h be the position of the first 1 in α_{s_k} after $\alpha_{s_k} \upharpoonright d_{e,s_{k-1}}$. At stage s_k we see $l_{e,s_k} = k$ and define M_e to wager 1 dollar on $(\alpha_{s_k} \upharpoonright h)^\wedge 1$. If α changes below $\gamma(e, k)[s_k - 1]$ at stage s_k then $\gamma(e, k)[s_k]$ will be set to at least h . Otherwise, $\gamma(e, k)[s_k - 1] \geq h$. Then as $k \geq i_0$, α changes below h at some stage $t' > s_k$. At stage t' , $\Gamma^\alpha(e, k)$ will become undefined. At the next *e-expansionary* stage we set $\Gamma^\alpha(e, k) = 1$ with use 0. This concludes the verification and the proof of Theorem 1.6.

3.2. Array computable c.e. degrees do not compute integer-valued randoms. A natural class of c.e. degrees that do not contain integer-valued randoms is the class of array computable degrees. In this section we sketch the proof of this fact, which along with Corollary 1.5 gives Theorem 1.7 that was presented in the introduction.

By [Ish99, TZ01] (also see [DH10, Proposition 2.23.12]) if A is array computable and c.e., h is a nondecreasing unbounded function and $f \leq_T A$, then there exists a computable approximation $(f[s])$ of f such that

$$(3.1) \quad |\{s \mid f(x)[s] \neq f(n)[s + 1]\}| \leq h(x) \text{ for all } x.$$

Hence given an integer-valued random B and a c.e. set A such that $B \leq_T A$, it suffices to define an order function h and a function $f \leq_T A$ such that any computable approximation $(f[s])$ to f does not satisfy (3.1). Let B be integer-valued random and suppose A is c.e. and $\Gamma^A = B$. We assume that at stage s , Γ has computed s many bits of $\Gamma^A[s]$. We define an order function h and a Turing functional Δ such that the function $f = \Delta^A$ does not satisfy (3.1) for any computable approximation $(f[s])$ of it. Let $\langle \psi_e \rangle$ be an effective list of all binary partial computable functions. We meet the requirements

$$\mathcal{R}_e: (\exists x)(f(x) \neq \lim_s \psi_e(x, s) \vee |\{s \mid \psi_e(x, s) \neq \psi_e(x, s + 1)\}| \geq h(x)).$$

We define for each $e \in \omega$ an integer-valued martingale m_e . First, let us describe the strategy for \mathcal{R}_0 . We will have $h(0) = 1$. At stage 1 we define $f(0) = 1$ with use $\delta_1(0) = \gamma_1(0)$. We wait until a stage s where we see $\psi_0(0, s) = f(0)[s] = 1$. If this happens, we will want to define m_0 to put pressure on A to change so that we may redefine $f(0)$. We define m_0 to start with \$1 in capital and wager \$1 on $\Gamma^A[s] \upharpoonright 1$. If $\Gamma^A \upharpoonright 1$ changes then we get a change in A below $\gamma_1(0)$, and so a change in A below $\delta_1(0)$. We may therefore redefine $f(0)$ and so meet \mathcal{R}_0 . We assume that there will be no change in $\Gamma^A \upharpoonright 1$, and so we immediately look to see whether we can start attacking \mathcal{R}_0 again by trying to redefine $f(1)$. At stage 2 we define $f(1) = 2$ with use $\delta_2(1) = \gamma_2(1)$. If we see no change in $\Gamma^A \upharpoonright 1$, then the martingale m_0 has \$2 on $\Gamma^A \upharpoonright 1$. We will have $h(1) = 1$. We wait until a stage s' where we see $\psi_0(0, s') = f(0)[s']$ and $\psi_0(1, s') = f(1)[s']$. If this happens, we define m_0 to wager \$1 on $\Gamma^A[s'] \upharpoonright 2$. If A changes below $\gamma_{s'}(1)$ then we may redefine $f(1)$ and meet \mathcal{R}_0 .

We would like m_0 to be total. Therefore whenever we let m_0 wager some of its capital on a string σ , we extend m_0 by letting it bet neutrally on all other strings of length at most $|\sigma|$. Now suppose that none of our previous attempts to redefine $f(0), \dots, f(x-1)$ have been successful. We wait until a stage s where we have $\psi_e(y, s) = f(y)[s]$ for all $y \leq x$. The use $\delta_s(x)$ will be equal to $\gamma_s(l)$ for some l . Suppose we have defined m_0 up to strings of length $l-1$ and that m_0 has $\$k$ on $\Gamma^A \upharpoonright (l-1)$. Suppose $h(x) = n$. Then we require n changes in A to redefine $f(x)$ as many times as we would like. If we let m_0 wager $\$1$ on $\Gamma^A \upharpoonright l$ and see A change below $\gamma_s(l)$, we can redefine $f(x)$ once. Suppose that we see this change at stage t . We lift $\delta_t(x) = \gamma_t(l+1)$. The martingale m_0 has been defined up to strings of length l , and we have $m_0(\Gamma_t^A \upharpoonright l) = k-1$. We again wait until a stage t' where $\psi_e(y, t') = f(y)[t']$ for all $y \leq x$. If this occurs, we now define m_0 to wager $\$2$ on $\Gamma_{t'}^A \upharpoonright (l+1)$. We do this so that if we do not see a change in this instance, m_0 's capital becomes $\$k+1$. When we set $\delta_{t'}(x) = \gamma_{t'}(l+1)$ this caused $f(x')$ to become undefined for all $x' > x$. At stage $t'+1$ we define $f(x+1) = x+2$ with use $\delta_{t'+1}(x+1) = \gamma_{t'+1}(l+2)$. Therefore, if necessary we may start attacking \mathcal{R}_0 by trying to redefine $f(x+1)$. If every time we see a change for $f(x)$ we increase our wager by $\$1$, after $n-1$ many changes we are left with $\$k - (1+2+\dots+n-1) = \$k - \frac{1}{2}(n-1)n$. In attempting to get the n th change, we wager all remaining capital and require that if we do not see another change, then we end up with more than $\$k$. So we want $2(k - \frac{1}{2}(n-1)n) > k$. That is, $k > (n-1)n$. We therefore set $h(0) = 1$ and let $h(n)$ be the greatest m such that $(m-1)m < h(n-1) + 1$. If we define m_0 as above then either we see all required changes, or m_0 's capital increases to at least $k+1$. As B is integer-valued random, we eventually do see all changes to redefine some $f(x)$, and satisfy \mathcal{R}_0 .

3.2.1. Multiple requirements and interactions. In order to deal with multiple requirements, we proceed as follows. The function $f = \Delta^A$ is a global object which must be defined on all inputs. As in the strategy above, the values $f(x)$ are changed by the action of the requirements. Suppose we satisfy \mathcal{R}_0 by redefining $f(0)$ once. We could attempt to satisfy \mathcal{R}_1 by further redefining $f(0)$, but at some point we must stop. We choose a fresh large number x_1 , and have the strategy for \mathcal{R}_1 try to redefine $f(x_1)$ as many times as necessary. As we saw above, the strategy for \mathcal{R}_1 may at any one time be wanting to redefine $f(y)$ for possibly many y . We formalise this by associating to each requirement \mathcal{R}_e at stage s an interval $I_{e,s}$ of natural numbers, so that \mathcal{R}_e at stage s is wanting to redefine $f(x)$ for $x \in I_{e,s}$. When we are successful in redefining $x \in I_{e,s}$, we remove all $y > x$ from $I_{e,s}$. If we have not already satisfied \mathcal{R}_e at some later stage s' and we see $\psi_e(z, s') = f(z)[s']$ for all $z \leq x+1$, then we add $x+1$ to $I_{e,s'}$ and attempt to redefine $f(x+1)$ as well.

Consider the requirements \mathcal{R}_e and \mathcal{R}_f , with \mathcal{R}_e of stronger priority than \mathcal{R}_f . We are defining martingales m_e for \mathcal{R}_e and m_f for \mathcal{R}_f . It is possible that when Γ^A moves and we redefine some $f(k)$ for the sake of \mathcal{R}_e that the martingale m_f also loses capital, even though we do not redefine some $f(j)$ for the sake of \mathcal{R}_f . We will therefore want to start a new version of m_f every time a requirement of stronger priority than \mathcal{R}_f acts. We say that \mathcal{R}_e requires attention at stage s if one of the following holds:

- (1) $I_{e,s} = \emptyset$.
- (2) for all $x \leq \max I_{e,s}$ we have $\psi_e(x, s) = f(x)[s]$ and $|\{t < s : f(x)[t] \neq f(x)[t+1]\}| < h(x)$, and $A_s(z) \neq A_{s-1}(z)$ for some $z \in (\delta_s(\min I_{e,s} - 1), \delta_s(\max I_{e,s}))$.
- (3) for all $x \leq \max I_{e,s}$ we have $\psi_e(x, s) = f(x)[s]$ and $|\{t < s : f(x)[t] \neq f(x)[t+1]\}| < h(x)$, and $\psi_e(\max I_{e,s} + 1, s) = f(\max I_{e,s} + 1)[s]$.

We are ready to produce the construction.

3.2.2. Construction. At stage 0, define $m_e(\lambda) = 1$ for all $e \in \omega$. Let $f(x)[0] = \Delta^A(x)[0] = 1$ with use $\delta_0(x) = x$ for all $x \in \omega$. Let $I_{e,1} = \emptyset$ for all $e \in \omega$. Each stage of the construction after stage 0 consists of three steps. At stage $s, s \geq 1$ proceed as follows:

Step 1: For all $e \leq s$, if a requirement of stronger priority than \mathcal{R}_e has acted since \mathcal{R}_e last acted, we start a new version of m_e , and define $m_e(\lambda) = 1$. Otherwise, we continue with the previous version of m_e . Let $d_{e,s}$ denote the length of the longest string for which the current version of m_e is defined.

Step 2: Let x be least such that $f(x)$ is undefined at the beginning of stage s . (If there is no such x , proceed to the next step.) Let $l = \max_{e \leq s} d_{e,s}$. Define $f(x)[s] = s+1$ with use $\delta_s(x) = \gamma_s(l+1)$.

Step 3: Let \mathcal{R}_e be the requirement of strongest priority which requires attention at stage s . Choose the first case by which \mathcal{R}_e requires attention.

If case 1 holds, choose a fresh large number x_e and let $I_{e,s+1} = \{x_e\}$.

If case 2 holds, then let $x \in I_{e,s}$ be least such that $A_s(z) \neq A_{s-1}(z)$ for some $z \in (\delta_s(x-1), \delta_s(x)]$. Let $f(x) = s+1$ with use $\delta_{s+1}(x) = \gamma_{s+1}(d_{e,s} + 1)$. We have that $m_e(\Gamma^A[s] \upharpoonright d_{e,s}) < m_e(\Gamma^A[s] \upharpoonright (d_{e,s} - 1))$. If $m_e(\Gamma^A[s] \upharpoonright d_{e,s}) \neq 0$, let $n_e = \max_{i \leq d_{e,s}} h(m_e(\Gamma^A[s] \upharpoonright i))$. Suppose that j is such that $n_e = h(m_e(\Gamma^A[s] \upharpoonright j))$ and let $n'_e = |\{m_e(\Gamma^A[s] \upharpoonright i) : j \leq i \leq d_{e,s}\}|$. Then we have received n'_e of the n_e permissions required to redefine $f(x)$ at least $h(x)$ many times. If $n'_e = n_e - 1$, then define m_e to wager $\Gamma^A[s] \upharpoonright d_{e,s}$ dollars on $\Gamma^A[s] \upharpoonright (d_{e,s} + 1)$. Otherwise let $w = m_e(\Gamma^A[s] \upharpoonright (d_{e,s} - 1)) - m_e(\Gamma^A[s] \upharpoonright d_{e,s})$ and define m_e to wager $\$(w + 1)$ on $\Gamma^A[s] \upharpoonright (d_{e,s} + 1)$. If $m_e(\Gamma^A[s] \upharpoonright d_{e,s}) = 0$, let m_e bet neutrally on all other strings of length $d_{e,s} + 1$. Let $I_{e,s+1} = [\min I_{e,s}, x]$.

If case 3 holds, then let $I_{e,s+1} = I_{e,s} \cup \{\max I_{e,s} + 1\}$. Define m_e to wager \$1 on $\Gamma^A[s] \upharpoonright (d_{e,s} + 1)$.

In any case, let $I_{e',s+1} = \emptyset$ for all $e' > e$.

3.2.3. Verification. We need to show that for all $e \in \omega$, \mathcal{R}_e is satisfied. Assume by induction that stage s^* is the last stage at which a requirement of stronger priority than \mathcal{R}_e acts. Assume for all $s \geq s^*$ that $\psi_e(x, s) = f(x, s)$ for all $x \leq \max I_{e,s}$. At stage $s^* + 1$ we will define a new version of m_e , which will be the final version. At every stage after $s^* + 1$, we define more of m_e . Therefore m_e is total. As Γ^A is integer-valued random, $m_e(\Gamma^A) = \sup \{m_e(\Gamma^A \upharpoonright i) : i \in \omega\} < \infty$. Let $\sup \{m_e(\Gamma^A \upharpoonright i) : i \in \omega\} = k$ and i_0 be such that $m_e(\Gamma^A \upharpoonright i_0) = k$. Suppose s_0 is least such that $s_0 \geq s^* + 1$ and $\Gamma^A[s_0] \upharpoonright i_0 = \Gamma^A \upharpoonright i_0$, and x is such that $\delta_{s_0}(x) = \gamma_{s_0}(d_{e,s_0} + 1)$. Then $f(x)$ is redefined $h(x)$ many times and \mathcal{R}_e is satisfied.

4. JUMP INVERSION FOR INTEGER-VALUED RANDOMS

Jump inversion for Martin-Löf randoms was discovered in [Kuč85, DM06] and was generalized in [BDN11]. Every degree which is c.e. in and above $\mathbf{0}'$ contains the jump of some Martin-Löf random Δ_2^0 set. Hence the same holds for the integer-valued randoms. However in this case we can obtain a stronger jump inversion theorem by requiring that the ‘inverted’ degrees are c.e. Note that this stronger theorem does not hold for Martin-Löf randoms since $\mathbf{0}'$ is the only c.e. degree containing a Martin-Löf random. Moreover it does not hold for computable randomness or Schnorr randomness, since by [NST05] the only c.e. degrees that contain such randoms are high. integer-valued randomness is the strongest known randomness notion for which jump inversion holds with c.e. degrees.

Since the argument is somewhat involved, we present it in two steps. In Section 4.1 we discuss the strategy for controlling the jump of an integer-valued random of c.e. degree. This argument gives a low c.e. degree which contains an integer-valued random. It is a finite injury construction, and the hardest of the two steps. Our argument actually shows the stronger result that there is a low c.e. weak truth table degree which contains an integer-valued random. We can then add coding requirements in order to prove the full jump inversion theorem, which we present in full detail in Section 4.2. This construction is a tree argument which uses the strategies of Section 4.1 for ensuring that the jump of the constructed set is below the given Σ_2^0 set, combined with standard coding requirements which deal with the remaining requirements.

4.1. A low c.e. degree containing an integer-valued random. We build an integer-valued random A of low c.e. degree. In fact, we build an integer-valued random A and a c.e. set B such that $A \equiv_{\text{wtt}} B$. Let $\langle m_e \rangle$ be an effective list of all partial integer-valued martingales. In order to ensure that A is integer-valued random it suffices to satisfy the following requirements:

- R_e : if m_e is total, then m_e does not succeed on A ;
- N_e : $(\exists^\infty s) (\Phi_e^A(e)[s] \downarrow) \implies \Phi_e^A(e) \downarrow$.

We order the requirements as $R_0 > N_0 > R_1 > N_1 > \dots$ and begin by setting $A_1 = 1^\omega$. To meet R_0 , we observe the values of the martingale m_0 . If m_0 increases its capital along A , we change A to force m_0 to lose capital. As m_0 is integer-valued, if it loses capital, it must lose at least \$1. Thus if we can force m_0 to lose capital every time we act, we need only act for R_0 finitely many times. As we are building reductions Γ and Δ such that $\Gamma^B = A$ and

$\Delta^A = B$, to change A we will need to change B . Once we have changed B , we will then need to change A again to record this fact. To satisfy the requirement N_e we use the usual strategy of preserving the restraint $\varphi_e^A(e)[s]$ at all but finitely many stages. As the strategy for an R -requirement is finitary, this can be done easily.

4.1.1. The finite injury construction. In order to help with the definition of the reductions, we make use of levels $\langle l_i \rangle_{i < \omega}$ and $\langle d_i \rangle_{i < \omega}$. We calculate the size of the levels below. We set $\gamma(l_i) = d_i$ and $\delta(d_i) = l_{i+1}$. We say that we act for requirement R_e at level l_{i+1} at stage s if we change A to decrease m_e 's capital from $A \upharpoonright l_i$ to $A \upharpoonright l_{i+1}$. That is, $m_e(A_s \upharpoonright l_{i+1}) > m_e(A_s \upharpoonright l_i)$ and $m_e(A_{s+1} \upharpoonright l_{i+1}) < m_e(A_{s+1} \upharpoonright l_i)$. We act at level l_{i+1} only for the sake of the requirements R_0, \dots, R_i . Once we have acted at level l_{i+1} , we enumerate an element from $[d_i, d_{i+1})$ into B . To record this change in B , we let A extend a string of length l_{i+2} which has not yet been visited. So that the reduction Δ is consistent, we must not let A extend a string which is *forbidden*, that is, a string σ such that $\Delta^\sigma \neq B_{s+1}$. We carefully define the levels l_i so that the action from the requirements never forces us to extend a forbidden string.

Before we define $\langle l_i \rangle_{i < \omega}$ and $\langle d_i \rangle_{i < \omega}$, we lay out the construction (in terms of these unspecified parameters and a function d defined below). In Section 4.1.2 we discuss the various properties that these parameters need to satisfy in order that the construction is successful (i.e. produces sets A, B which satisfy the requirements R_e, N_e).

We have for every $e \in \omega$ and every stage s a restraint $r_{e,s}$. We say that R_e requires attention at level l_{i+1} at stage s if

- (1) $m_e(\sigma)[s] \downarrow$ for all strings σ of length $\leq l_{i+1}$,
- (2) $l_{i+1} \geq r_{e,s}$
- (3) $m_e(A_s \upharpoonright l_{i+1}) > m_e(A_s \upharpoonright l_i)$.

We say that R_e requires attention at stage s if it does so at some level. We say that N_e requires attention at stage s if $\Phi_e^A(e)[s] \downarrow$. Recall the definition of the *sibling* of a string, just after Definition 1.1.

Construction. Let $\gamma(l_i) = d_i$ and $\delta(d_i) = l_{i+1}$. At stage 0, let $A_1 = 1^\omega$ and $r_{e,1} = l_e$ for all e .

Stage $s, s \geq 1$: Find the requirement of strongest priority which requires attention at stage s . (If there is no such requirement, proceed to the next stage.)

Case 1: If this is R_e , let l_{i+1} be least such that R_e requires attention at level l_{i+1} at stage s . Let $l \in (l_i, l_{i+1}]$ be least such that $m_e(A_s \upharpoonright l) > m_e(A_s \upharpoonright l_i)$. Choose a string τ of length l_{i+1} which extends the sibling of $A_s \upharpoonright (l-1)$ such that the minimum of all $d(\tau, \mu)$, where μ is any forbidden string of length l_{i+1} extending $A_s \upharpoonright l_i$, is as large as possible. Enumerate an element of $[d_i, d_{i+1})$ into B . Choose a string ρ of length l_{i+2} extending τ such that $\rho \neq A_t$ for all $t < s$, and the minimum of all $d(\rho, \mu)$, where the minimum is taken over forbidden strings μ of length l_{i+2} extending τ , is as large as possible. Set $A_{s+1} = \rho 1^\omega$. For all $e' \geq e$ with $r_{e',s} \leq l_{i+1}$, let $r_{e',s+1} = l_{i+1}$.

Case 2: If this is N_e , for all $e' > e$ with $r_{e',s} \leq \varphi_e^A(e)[s]$, let $r_{e',s+1} = \varphi_e^A(e)[s]$.

In the following section we give the remaining specifications and analysis of the construction, as well as the verification.

4.1.2. The calculation of the levels l_i and d_i for a successful construction. In the following we calculate the levels l_i, d_i , and depict this process in Figure 2. Suppose that we act at level l_{i+1} for R_e and naively let A extend a string τ of length l_{i+1} whose sibling is forbidden. Consider the situation where m_0 increases its capital on the very last bit of $A_s \upharpoonright l_{i+1}$, loses capital on τ , and is neutral on all other strings of length l_{i+1} . We will not be able to change A to extend τ 's sibling, as this string is forbidden. However, we do not want to change A so that m_0 is neutral, as we would like the action for R_0 to be finitary. To avoid such a situation we must be more careful in how we change A . In particular, we must ensure that A is kept in some sense “far away” from forbidden strings. This is made precise below.

We first calculate an upper bound on the number of forbidden strings of length l_{i+1} which can occur above a nonforbidden string of length l_i . Our upper bound will not be strict. A string σ of length l_{i+1} becomes forbidden if Δ^σ is no longer giving correct B -information. As $\delta(d_i) = l_{i+1}$, Δ^σ will be incorrect only if we enumerate an element into B below d_i , which occurs only when we act for a requirement at some level $\leq l_i$. We will act for R_e at level l_{i+1} only when we see m_e halt on all strings of length l_{i+1} , and so if A no longer changes below l_i , we will act for R_e at

level l_{i+1} at most once. As we act at level l_{i+1} only for the sake of requirements R_0, \dots, R_i , if A no longer changes below l_i , we can act at level l_{i+1} at most $i + 1$ times. After acting at a level l_j for some $j \leq i$, we allow R_0, \dots, R_i to act at level l_{i+1} again. We begin with $A_1 = 1^\omega$. Suppose we act $i + 1$ times at level l_{i+1} . We then act at level l_i . We act another $i + 1$ times at level l_{i+1} before we again act at level l_i . We can act at level l_i at most i times. This can continue until we get to level l_1 , where we can change A once below l_1 for the sake of requirement R_0 . Therefore we act at a level $\leq l_i$ $2 \cdot 3 \cdot 4 \cdot \dots \cdot (i + 2) = (i + 2)!$ many times, and there are at most

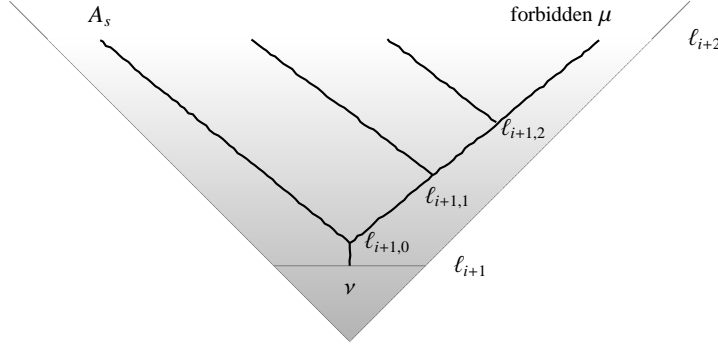
$$f_{i+1} = \sum_{j=0}^i (j + 1)!$$

many forbidden strings of length l_{i+1} . Note that for any $k \in \omega$ we may enumerate all partial integer-valued martingales with initial capital k . We therefore may assume that our list $\langle m_e \rangle$ of all partial integer-valued martingales comes with a computable initial capital, $m_e(\lambda)$. As a martingale may at most double its capital in a single bet, the upper bound on m_e 's capital at a string of length n is $2^n m_e(\lambda)$.

We now show how a martingale can force us "closer" to a forbidden string. Suppose at stage s that A_s extends the string ν of length l_{i+1} , and there is a forbidden string μ of length l_{i+2} above. For simplicity, suppose that A_s extends the leftmost string of length l_{i+2} which extends ν , and that μ is the rightmost string of length l_{i+2} which extends ν . If \mathcal{R}_j requires attention at level l_{i+2} , we would like to choose a string τ of length l_{i+2} with $m_j(\tau) < m_j(\nu)$. The problem is the following. Suppose that m_j increases its capital on all string of length l_{i+2} which extend $\nu 0$. By Kolmogorov's inequality, this must mean that $m_j(\nu 0) > m_j(\nu)$, and so $m_j(\nu 1) < m_j(\nu)$. If m_j has sufficient capital at $\nu 1$, it may then increase its capital above $m_j(\nu)$ on all strings of length l_{i+2} which extend $\nu 1 0$. Again by Kolmogorov's inequality we have $m_j(\nu 1 0) > m_j(\nu 1)$ and so $m_j(\nu 1 1) < m_j(\nu 1) < m_j(\nu)$. Now m_j is an integer-valued martingale, and so after doing this finitely many times, say n times, we have $m_j(\nu 1^n) < \frac{1}{2} m_j(\nu)$ and so m_j cannot increase its capital above $m_j(\nu)$ on all strings of length l_{i+2} which extend $\nu 1^n 0$. If $l_{i+2} \geq l_{i+1} + n + 1$, then we can pick a string τ extending $\nu 1^n 0$ with $m_j(\tau) < m_j(\nu)$ and which is not forbidden. For two strings α and β of length l , let $d(\alpha, \beta)$ be $l - b$, where b is the length of the longest common initial segment. Then in this situation, we have $d(A_s, \mu) = l_{i+2} - l_{i+1}$ and $d(\tau, \mu) \leq l_{i+2} - l_{i+1} - n$. Therefore m_j has forced A distance n closer to the forbidden string μ . Now \mathcal{R}_j might not be the only requirement which can act at level l_{i+2} . We will then need to calculate the distance that the other martingales may move A , and ensure that l_{i+2} is high enough.

We calculate a bound on how far an integer-valued martingale m may move A . If $m(\nu) = k$ and m increases its capital to $k + 1$ on all strings extending $\nu 0$, then $m(\nu 1) = k - 1$. If $m(\nu 1) \geq 2$ then m can increase its capital to $k + 1$ on all strings extending $\nu 1 0$. Then $m(\nu 1 1) = k - 1 - 2$. If $m(\nu 1) \geq 4$ then m can increase its capital to $k + 1$ on all strings extending $\nu 1 1 0$. Then $m(\nu 1 1 1) = k - 1 - 2 - 4$. When $m(\nu 1^n) = k - 1 - 2 - \dots - 2^{n-1} < \frac{1}{2}k$, m is not able to increase its capital to $k + 1$ on all strings extending $\nu 1^n 0$. We let $n(k) = (\mu n)(k - 1 - 2 - \dots - 2^{n-1} < \frac{1}{2}k)$. Then m can move A at most a distance $n(k)$. In the case where A_s is not the leftmost string extending ν and μ is not the rightmost string extending ν , a similar argument shows that m_e can still move A a distance of at most $n(k)$. The only difference is that m_e would then need to bet against the initial segments of μ which are of length greater than $l_{i+2} - l_{i+1} - d(A_s, \mu)$.

Suppose we act at level l_{i+1} at stage s and let A extend the string ν of length l_{i+1} which has forbidden strings of length l_{i+2} above. We enumerate an element of $[d_i, d_{i+1})$ into B . To record this change in B , we choose a string ρ of length l_{i+2} which has not been visited before, and which is as far from any forbidden string as possible. We have that there are at most f_{i+2} many forbidden strings of length l_{i+2} above a nonforbidden string of length l_{i+1} . Let $x = (\mu x)(2^x \geq f_{i+2})$. By the counting argument for f_{i+1} given above, if $l_{i+2} - l_{i+1} = h > x$, then there is a string of length l_{i+2} which has not been visited yet, and which is at least distance $h - x$ from a forbidden string. Now suppose that \mathcal{R}_j requires attention at level l_{i+2} . We know that there are no forbidden strings above $\rho \upharpoonright l_{i+1} + x + 1$, and so if we can reduce m_j by moving to a string which is still above $\rho \upharpoonright l_{i+1} + x + 1$, we will do so. Otherwise, m_j will move us closer to a forbidden string. The bound on the capital of m_j at $A_s \upharpoonright (l_{i+1} + x + 1)$ is $2^{l_{i+1}+x+1} m_j(\lambda)$. So we know that m_j may move us a distance at most $n(2^{l_{i+1}+x+1} m_j(\lambda))$ towards a forbidden string. If $l_{i+2} - l_{i+1} > x + n(2^{l_{i+1}+x+1} m_j(\lambda))$ then we will be able to choose a nonforbidden string ρ' which decreases m_j . Suppose that \mathcal{R}_k , which is of stronger priority than \mathcal{R}_j , requires attention at level l_{i+2} at stage s' . Our reasoning

Figure 2. Calculating the levels l_i and avoiding the forbidden strings.

is similar to the previous case. Let $n_0 = n(2^{l_{i+1}+x+1}m_j(\lambda))$. We know that there are no forbidden strings above $\rho' \upharpoonright l_{i+1} + x + n_0 + 1$, and so if we can reduce m_k by moving to a string which is still above $\rho' \upharpoonright l_{i+1} + x + n_0 + 1$, we will do so. Otherwise, m_k will move us closer to a forbidden string. The bound on the capital of m_k at $A_{s'} \upharpoonright (l_{i+1} + x + n_0 + 1)$ is $2^{l_{i+1}+x+n_0+1}m_k(\lambda)$. So we know that m_k may move us a distance at most $n(2^{l_{i+1}+x+n_0+1}m_k(\lambda))$ towards a forbidden string. If $l_{i+2} - l_{i+1} > x + n_0 + n(2^{l_{i+1}+x+n_0+1}m_k(\lambda))$ then we will be able to choose a nonforbidden string ρ'' which decreases m_k . We will need l_{i+2} to be large enough so that we can always move in this way for any requirement which might act at level l_{i+2} .

The requirements $\mathcal{R}_0, \dots, \mathcal{R}_i$ can act at level l_{i+1} . We do not know the order in which the requirements may act, so we will have to take the maximum of the capitals of the martingales m_0, \dots, m_i in our calculation. We illustrate this definition in Figure 2. We set $l_0 = 0$. Given l_i , we set $l_{i+1,0} = l_i + (\mu x)(2^x \geq f_{i+1})$ and for $0 \leq j \leq i$,

$$l_{i+1,j+1} = l_{i+1,j} + \max_{k \leq i} n(2^{l_{i+1,j}+1}m_k(\lambda))$$

and let $l_{i+1} = l_{i+1,i+1} + 1$. The levels d_i are chosen so that we can enumerate an element into $[d_i, d_{i+1})$ every time we act at level l_{i+1} . This calculation is the same as that of f_{i+1} . Let $d_0 = 0$. Given d_i , let $d_{i+1} = d_i + \sum_{j=0}^i (j+1)!$.

4.1.3. Verification of the finite injury construction. First, we show that $A \equiv_{\text{wtt}} B$. As in the calculation of f_{i+1} , we can act at level l_{i+1} at most $\sum_{j=0}^i (j+1)!$ many times. We have that $d_{i+1} = d_i + \sum_{j=0}^i (j+1)!$ for all i . Every time we act at level l_{i+1} and change A below l_{i+1} , we enumerate an element from $[d_i, d_{i+1})$ into B . The uses $\gamma(l_i)$ are clearly computable, and so we have $\Gamma^B = A$ via the weak truth-table functional Γ . For the other reduction, note that the consistency of Δ is a consequence of A never extending a forbidden string. Again the uses $\delta(d_i)$ are computable and so $\Delta^A = B$ via the weak truth-table functional Δ .

Next we argue that all N_e requirements are met. Suppose inductively that all requirements of stronger priority than N_e do not act after a certain stage t . If at some stage s after stage t the computation in requirement N_e halts, then a restraint $r_{e,s}$ will be erected so that the use of the computation is protected from further enumerations into A . Therefore in that case the computation actually halts. Therefore N_e is met, and this concludes the induction step.

It remains to show that for every $e \in \omega$, \mathcal{R}_e is satisfied. Suppose by induction that all requirements of stronger priority than \mathcal{R}_e do not act after stage s^* . Let i_0 be least such that $l_{i_0} \geq r_{e,s^*}$. We show that $m_e(A) \leq m_e(A_{s^*} \upharpoonright l_{i_0})$. Suppose at stage $s \geq s^*$ we see m_e increase its capital beyond $m_e(A_{s^*} \upharpoonright l_{i_0})$. Then \mathcal{R}_e will require attention at stage s . Suppose that \mathcal{R}_e requires attention at level l_{i+1} at stage s . Let $t < s$ be the last stage at which we acted for some requirement at a level below l_{i+1} . Suppose we acted at level l_{j+1} . At stage t we chose some string ρ of length l_{j+2} and let $A_t = \rho 1^\omega$. Then $A_{s-1} \upharpoonright l_i = A_t \upharpoonright l_i$. If $j = i - 1$ then we chose ρ which would have been at least distance $l_{i+1} - l_i - x$, where $x = (\mu x)(2^x \geq f_{i+1})$, from any forbidden string of length l_{i+1} (that is, ρ and any forbidden string have a common initial segment of length at most $l_i + x$). Otherwise $j < i - 1$ and there is no forbidden string of length l_{i+1} above $A_s \upharpoonright l_i$. Suppose that between stages t and $s - 1$ inclusive we acted at level l_{i+1} k many times. We

have that $k < i + 1$. Then $A_s \upharpoonright l_i$ and any forbidden string have a common initial segment of length at most $l_{i+1,k}$. Let $l \in (l_i, l_{i+1}]$ be least such that $m_e(A_s \upharpoonright l) > m_e(A_s \upharpoonright l_i)$. If $l > l_{i+1,k} + 1$, then there is a string τ above $A_s \upharpoonright l_{i+1,k} + 1$ with $m_e(\tau) < m_e(A_s \upharpoonright l)$ and which is not forbidden. Otherwise m_e can move us at most distance $n(2^{l_{i+1,k}+1}m_e(\lambda))$ closer to a forbidden string. We have that $l_{i+1} > l_{i+1,k} + n(2^{l_{i+1,k}+1}m_e(\lambda))$, so we can find a nonforbidden string τ with $m_e(\tau) < m_e(A_s \upharpoonright l)$. Restraints are then imposed so that \mathcal{R}_e and no other requirement of weaker priority may act at level l_{i+1} after stage s . Therefore $A \upharpoonright l_{i+1} = A_{s+1} \upharpoonright l_{i+1}$ and so $m_e(A) < m_e(A_{s^*} \upharpoonright l_{i_0})$.

4.2. The full jump inversion theorem for integer-valued randoms. Given a set $S \geq_T \mathbf{0}'$ which is c.e. in $\mathbf{0}'$ we show how to construct an integer-valued random set A of c.e. degree such that $A' \equiv_T S$. Along with A , we build a c.e. set B such that $A \equiv_T B$, and show that $B' \equiv_T S$. Let $\langle m_e \rangle$ be an effective enumeration of all partial computable integer-valued martingales. So that A is integer-valued random, we meet the requirements

\mathcal{R}_e : If m_e is total, then m_e does not succeed on A .

We also build wtt-reductions Γ and Δ such that $\Gamma^B = A$ and $\Delta^A = B$. For the requirement $S \leq_T B'$, we build a functional Λ and meet the requirements

\mathcal{P}_e : $\lim_t \Lambda^B(e, t) = S(e)$.

The basic strategy for a \mathcal{P} -requirement is as follows. As S is c.e. in and above $\mathbf{0}'$, we know that S is Σ_2^0 . Therefore there is some computable approximation $\{S_i\}_{i \in \omega}$ such that $n \in S$ if and only if there is an s such that $n \in S_i$ for all $t > s$. We define $\Lambda^B(e, s) = 1$ for larger and larger s with some large use $\lambda(e, s)$. If we see at some stage u that $e \notin S_u$ and $u > t$, then we enumerate $\lambda(e, t)$ into B and redefine $\Lambda^B(e, t) = 0$ with use -1 , i.e. the axiom defining $\Lambda^B(e, t) = 0$ does not depend on B .

For the requirement $B' \leq_T S$ we attempt to meet the requirements

\mathcal{N}_e : $(\exists^\infty s)(\Phi_e^B(e) \downarrow) \implies \Phi_e^B(e) \downarrow$.

We attempt to meet these as usual by restraining B below the use $\varphi_e^B(e)[s]$ whenever we see $\Phi_e^B(e)[s] \downarrow$. Although we will not actually meet these requirements (doing so would mean that $B' \equiv_T \mathbf{0}'$), trying to meet the requirements will allow us to show that $B' \leq_T S$.

4.2.1. The priority tree. The construction will use a tree of strategies. To define the tree, we specify recursively the association of nodes to requirements, and specify the outcomes of nodes working for particular requirements. To specify the priority ordering of nodes, we specify the ordering between outcomes of any node. We order the requirements as

$$\mathcal{R}_0 > \mathcal{P}_0 > \mathcal{N}_0 > \mathcal{R}_1 > \mathcal{P}_1 > \mathcal{N}_1 > \dots$$

and specify that all nodes of length k work for the k^{th} requirement on the list. We will have nodes dedicated to \mathcal{R} -, \mathcal{P} -, and \mathcal{N} -requirements. A node dedicated to a \mathcal{P} -requirement will have the ∞ outcome, corresponding to enumerating infinitely many markers $\lambda(e, s)$, and the f outcome, for when only finitely many markers are enumerated. Suppose that the node α works for \mathcal{P}_e and β works for \mathcal{N}_f with $\alpha < \beta$. If f is the true outcome of α and $\alpha \wedge f \leq \beta$, then only finitely many markers are enumerated, and β does not need to worry about the computations it sees being destroyed infinitely many times. Now suppose that ∞ is the true outcome of α and $\alpha \wedge \infty \leq \beta$. Then β will be guessing that \mathcal{P}_e will enumerate all its unrestrained markers $\lambda(e, s)$ into B . It will then not believe a computation $\Phi_e^B(e)[s]$ until it sees that all unrestrained markers below the use $\varphi_e^B(e)[s]$ have been enumerated. This is formalized with the definition of a β -correct computation below. The outcomes of \mathcal{R} - and \mathcal{N} -nodes are $\dots < n < \dots < 1 < 0$, corresponding to the restraint they impose on B .

4.2.2. Making the sets A, B of the same degree. In Section 4.1 we discussed the proof that there is a low c.e. degree containing an integer-valued random set. As in that argument, we make use of *levels* in the definition of the reductions Γ and Δ . We slightly adjust the definition of the levels because we now must also enumerate the markers $\lambda(e, s)$ into B . We increase the size of each interval $[d_i, d_{i+1})$ to accommodate a coding marker. The coding markers will be chosen to be d_i for some $i \in \omega$. Now that we are also enumerating coding markers into B , we also adjust the definition of the levels l_i . Enumerating the coding markers will cause more strings to become *forbidden*. We recalculate the upper bound on the number of forbidden strings of length l_{i+1} . As before, the requirements

R_0, \dots, R_i may act at level l_{i+1} . We calculated in Section 4.1 that we may act at most $(i+2)!$ many times at level l_{i+1} . When the coding marker d_i is enumerated, the requirements R_0, \dots, R_i may act again at level l_{i+1} . Therefore we may act at most $2(i+2)! + 1$ many times at level l_{i+1} . So there are at most $\sum_{j=0}^i (2(j+1)! + 1)$ many forbidden strings of length l_{i+1} . Letting $f'_{i+1} = \sum_{j=0}^i (2(j+1)! + 1)$, we calculate the levels l_i as before. We let $l_0 = 0$, and given l_i , we let $l_{i+1,0} = l_i + (\mu n)(2^n \geq f'_{i+1})$ and for $0 \leq j \leq i$ we let

$$l_{i+1,j+1} = l_{i+1,j} + \max_{k \leq i} n(2^{l_{i+1,j}+1} m_k(\lambda))$$

and $l_{i+1} = l_{i+1,i+1} + 1$. Set $d_0 = 0$. Given d_i , let $d_{i+1} = d_i + 2(i+1)! + 1 + 1$.

4.2.3. Coordination and restraint on the tree. The action for \mathcal{R} -requirements will otherwise be identical with the construction of Section 4.1. Nodes working for \mathcal{R} -requirements will also have to be wary of coding done by \mathcal{P} -nodes above. Suppose β , working for \mathcal{R}_f , is below the ∞ outcome of α , which is working for \mathcal{P}_e . If we see the martingale m_f increase its capital along A_s and wish to enumerate an element of $[d_i + 1, d_{i+1})$ into B to change A , we will wait until all unrestrained markers $\lambda(e, s)$ below d_i have been enumerated into B before changing A for the sake of \mathcal{R}_f . If σ is accessible at stage s , we let

$$r(\sigma, s) = \max\{r(\alpha) \mid \alpha < \sigma \text{ is an } \mathcal{N}\text{-node or an } \mathcal{R}\text{-node}\}.$$

We say that a computation $\Phi_e^B(e)[s]$ is σ -correct if for every \mathcal{P} -node α such that $\alpha^\wedge \infty \leq \sigma$, $r(\alpha, s) < \lambda(e, t) < \varphi_e^B(e)[s]$ implies $\lambda(e, t) \in B_s$. Recall the definition of the *sibling* of a string, just after Definition 1.1.

4.2.4. Construction of the sets A, B . At stage 0 we set $A_1 = 1^\omega$, $B_1 = \emptyset$, and let $r_{e,1} = l_e$ for all e . Moreover we set $\Lambda^B(0, 0) = 0$ with use d_1 . Each stage $s \geq 1$ is conducted in two steps:

Step 1: For $e, t \leq s$, if $\Lambda^B(e, t)$ is undefined at stage s , then let $\Lambda^B(e, t) = 1$ with some fresh large use $\lambda(e, t)$ equal to d_i for some $i \in \omega$.

Step 2: We let the collection of accessible nodes δ_s be an initial segment of the tree of strategies. Let σ be a node which is accessible at stage s . We describe the action that σ takes, and if $|\sigma| < s$, then we specify which immediate successor of σ is also accessible at stage s ; otherwise, we proceed to the next stage.

Suppose first that σ works for \mathcal{R}_e . Let k be least such that $d_k \geq r(\sigma, s)$. If

- (1) for all \mathcal{P} -nodes α such that $\alpha^\wedge \infty \leq \sigma$, $r(\alpha, s) < \lambda(e, t) < d_k \implies \lambda(e, t) \in B_s$, and
- (2) there is $l > l_{k+1}, r_{e,s}$ such that
 - (a) $m_{e,s}(\sigma) \downarrow$ for all strings σ of length l , and
 - (b) $m_e(A_s \upharpoonright l) > m_e(A_s \upharpoonright (l-1))$,

then let i be such that $l \in (l_i, l_{i+1}]$ (if there is more than one such l , we choose the least). Choose a string τ of length l_{i+1} which extends the sibling of $A_s \upharpoonright (l-1)$ such that the minimum of all $d(\tau, \mu)$, where μ is any forbidden string of length l_{i+1} extending $A_s \upharpoonright l_i$, is as large as possible. Enumerate an element of $[d_i + 1, d_{i+1})$ into B . Choose a string ρ of length l_{i+2} extending τ such that $\rho \neq A_t$ for all $t < s$, and the minimum of all $d(\rho, \mu)$, where the minimum is taken over forbidden strings μ of length l_{i+2} extending τ , is as large as possible. Set $A_{s+1} = \rho 1^\omega$. For all $e' \geq e$ with $r_{e',s} \leq l_{i+1}$, let $r_{e',s+1} = l_{i+1}$. The string $\sigma^\wedge d_{i+1}$ is accessible at stage s .

Now suppose that σ works for \mathcal{P}_e . For all $e, t \leq s$, if $\lambda(e, t) > r(\sigma, s)$, $e \notin S_s$ and we have $\Lambda^B(e, t)[s] = 1$, then enumerate $\lambda(e, t)$ into B and define $\Lambda^B(e, t) = 0$ with use -1 . Suppose that $\lambda(e, t) = d_i$. Choose a string ρ of length l_{i+1} extending $A_s \upharpoonright l_i$ such that $\rho \neq A_t$ for all $t < s$, and the minimum of all $d(\rho, \mu)$, where the minimum is taken over forbidden strings μ of length l_{i+1} extending $A_s \upharpoonright l_i$, is as large as possible. Set $A_{s+1} = \rho^\wedge 1^\omega$. If a marker was enumerated, let $\sigma^\wedge \infty$ be accessible at stage s . Otherwise let $\sigma^\wedge f$ be accessible at stage s .

Finally suppose that σ works for \mathcal{N}_e . If $\Phi_e^B(e)[s] \downarrow$ via a σ -correct computation, then let $\sigma^\wedge \varphi_e^B(e)[s]$ be accessible at stage s . Otherwise, let $\sigma^\wedge 0$ be accessible at stage s .

4.2.5. *Verification of the construction.* By the construction, the set B is c.e. We verify that $A \equiv_T B$, that A is ivr, and that $A' \equiv_T S$. First, we establish the existence of a ‘true path’.

(4.1) The leftmost path which is visited infinitely often exists.

As there are only finitely many outcomes of a \mathcal{P} -node, we need to verify that the restraint imposed by an \mathcal{N} - or \mathcal{R} -node comes to a limit. Let σ work for \mathcal{R}_e , and suppose by induction that no node to the left of σ is visited after stage s_0 , and that $\lim_s r(\sigma, s)$ exists. We must have $r(\sigma, s) = r(\sigma, s_0)$ and $r_{e,s} = r_{e,s_0}$ for all $s \geq s_0$. Let k be such that $d_k \geq r(\sigma, s_0)$. Whenever we act for \mathcal{R}_e at level l_i for $i \geq k+1$, m_e ’s capital decreases by at least \$1 and we increase restraints for all \mathcal{R} -requirements of weaker priority. The only elements which may be enumerated below the restraints \mathcal{R}_e places on B are coding markers belonging to \mathcal{P} -requirements stronger than \mathcal{R}_e . However, if \mathcal{R}_e is below the ∞ outcome of \mathcal{P}_e , then \mathcal{R}_e waits until all unrestrained markers below d_i enter B before acting at level l_{i+1} . Therefore the only markers which enter below \mathcal{R}_e ’s restraint belong to those \mathcal{P} -requirements with σ below the f outcome. By induction, we do not visit any node to the left of σ after stage s_0 , and so no such strategy may act after stage s_0 and enumerate a coding marker below \mathcal{R}_e ’s restraint. Therefore we act for \mathcal{R}_e only finitely many times after stage s_0 . Similarly, as we require the computations \mathcal{N} -nodes observe to be σ -correct, if σ works for \mathcal{N}_e and is on the true path, it will increase its restraints only finitely many times. This concludes the proof of (4.1).

Let the true path, TP, be the leftmost path visited infinitely often. The proof of (4.1) shows that we act only finitely often for any \mathcal{R} -requirement. Therefore $m_e(A) < \infty$ and \mathcal{R}_e is satisfied for all $e \in \omega$. Therefore

(4.2) the set A is integer-valued random.

The use of the systems of levels l_i, d_i in the construction define Turing reductions $A \leq_T B$ and $B \leq_T A$ respectively, with computable use. So that by an induction on the stages of the construction we have

(4.3) the sets A and B are in the same weak truth table degree.

It remains to show that $S \equiv_T B'$. First, we show that $S \leq_T B'$. Let $\sigma < \text{TP}$ be devoted to \mathcal{P}_e , and suppose that no node to the left of σ is visited after stage s_0 . As the restraints set by \mathcal{R} - and \mathcal{N} -requirements are finite, $r(\sigma, s_0)$ is finite and $r(\sigma, s) = r(\sigma, s_0)$ for all $s \geq s_0$. Therefore σ may enumerate all but finitely many markers if it wishes. Therefore for all $e \in \omega$, $\lim_t \Lambda^B(e, t) = S(e)$. It remains to show that $B' \leq_T S$. We have

$$(4.4) \quad e \in B' \Leftrightarrow \Phi_e^B(e) \downarrow \Leftrightarrow (\exists t)(\Phi_e^B(e)[t] \downarrow \wedge B \upharpoonright \varphi_e^B(e)[t] = B_t \upharpoonright \varphi_e^B(e)[t]).$$

First use S to compute $S \upharpoonright e$. For $i < e$, if $S(i) = 1$ then we will want to eventually stop enumerating markers $\lambda(i, s)$. If $S(i) = 0$, then we will want to enumerate all unrestrained markers. Suppose we see a computation $\Phi_e^B(e)[t] \downarrow$. We find the markers $\lambda(i, s)$ below $\varphi_e^B(e)[t]$ for $i < e$. As we know the fate of every marker below the use, we can computably determine whether this computation is B -correct, that is, whether B will change below the use after stage t . Therefore equation (1) is Σ_1^0 , and can be decided by \emptyset' . As $\emptyset' \leq_T S$, we have $B' \leq_T S$.

4.3. Degrees of left-c.e. partial integer-valued randoms. Here we show that every left-c.e. real that is integer-valued random is Turing (and in fact, weak truth table) equivalent to a partial integer-valued random left-c.e. real. Hence along with the argument of the previous section, it proves Theorem 1.6. In order to make the argument more concise, we will often refer to the argument of Section 4.1, which uses a similar machinery. Given a left-c.e. integer-valued random set A we will construct a partial integer-valued random set B such that $A \equiv_{\text{wtt}} B$. Suppose we are given A with a computable approximation $\langle A_s \rangle$. Let $\langle \varphi_e \rangle$ be an effective enumeration of all partial computable integer-valued martingales. We build a set B and weak truth-table reductions Γ and Δ such that $\Gamma^A = B$ and $\Delta^B = A$ to meet the requirements

\mathcal{R}_e : φ_e does not succeed on B .

We also build for each $e \in \omega$ a partial integer-valued martingale m_e . In the case that φ_e is defined along B , m_e will be total.

4.3.1. *Strategy for the single requirement \mathcal{R}_0 .* Let $\gamma(n)[s]$ be the use of computing $\Gamma^A(n)[s]$ and $\delta(n)[s]$ the use of computing $\Delta^B(n)[s]$. We begin by setting $\gamma(n)[0] = n$ and $\delta(n)[0] = n$ for all n . We observe the values of φ_0 along B . First we wait to see $\varphi_0(\lambda)$. If we later see φ_0 increase its capital along B , then we will wish to change B to force φ_0 to lose capital. We will need permission from A to do so. We put pressure on A to change by defining the martingale m_0 . If φ_0 increases its capital on B_s by betting on $B_s \upharpoonright n$, then we define m_0 to start with capital $\varphi_0(\lambda)$, place the same bets as $\varphi_0[s]$ along $B_s \upharpoonright n$, and bet neutrally on all other strings up to length n . We repeat this every time we see φ_0 increase its capital along B until we see a change in A . As A is integer-valued random, $m_0(A) < \infty$ and so A must eventually move. This gives us a permission to change B .

Suppose that at stage s we have defined m_0 up to length d , and A changes below d . Let m be least such that $A_{s-1}(m) \neq A_s(m)$. We have defined $\delta(m)[s-1] = m$ and so we must change B on its m th bit. We let $B_s = B_{s-1} \upharpoonright m \wedge (1 - B_{s-1}(m)) \wedge 0^\omega$. The partial martingale φ_0 might not be defined on any string extending $B_s \upharpoonright m$, whereas m_0 has been defined to be neutral on all initial segments of B_s of length between m and d . Later φ_0 might increase its capital along these strings, and we would not be able to define m_0 to directly copy its bets. We can however raise the use $\gamma(m)[s]$ to be large. At stage s we let $\gamma(m)[s] = 2d$. If φ_0 then bets along initial segments of B of length between m and d , we copy the wagers that φ_0 makes on these strings by placing the same wagers along the initial segments of A of length between d and $2d$. We are then still putting pressure on A to change. If A changes below $2d$ we can then change B below d and force φ_0 to lose money.

4.3.2. *Multiple requirements.* The interaction between multiple requirements will cause difficulty in coding. We use levels $\langle l_i \rangle$ and $\langle d_i \rangle$ in order to facilitate the coding. We set $\gamma(l_i) = d_i$ and $\delta(d_i) = l_{i+1}$ and let $l_1 = 5$ (the choice of l_1 is not significant). We attempt to change B above a string of length l for $l \in (l_i, l_{i+1}]$ only for the sake of the requirements $\mathcal{R}_0, \dots, \mathcal{R}_i$.

We will attempt to change B below l_1 only for the sake of decreasing φ_0 's capital along $B \upharpoonright l_1$. We would like $\gamma(l_1)$ to be large enough so that we can copy all the wagers that φ_0 may place along strings of length l_1 . There are 2^{l_1} many such strings, and so if we set $d_1 = \gamma(l_1) = 2^{l_1} \cdot l_1$, this will certainly be large enough. As A is left-c.e., A can change below d_1 at most 2^{d_1} many times. Therefore there are at most 2^{d_1} many forbidden strings. To calculate l_2 , we begin by setting $l_{2,0} = l_1 + (\mu x)(2^x \geq 2^{d_1}) = l_1 + d_1$. We act at level l_2 for the sake of \mathcal{R}_0 and \mathcal{R}_1 . The action for these requirements can again move us closer to forbidden strings. The distance we can be moved, is given in terms of the function d which is introduced in the argument of Section 4.1. Therefore we define $l_{2,1}, l_{2,2}$ and l_2 as before.

Suppose A changes below d_1 at stage s . Then we are free to change B below l_1 . We choose a string τ of length l_1 which minimises m_0 ; if there is no reason to move, we do not move. In either case, we then choose some string ρ of length l_2 extending the current version of $B \upharpoonright l_1$ which has not been visited before, and let $B_{s+1} = \rho 1^\omega$.

We change B below l_2 for the sake of requirements \mathcal{R}_0 and \mathcal{R}_1 . We define the total martingale m_0 to copy the wagers that φ_0 places on $B \upharpoonright l_2$, and we define the total martingale m_1 to copy the wagers that φ_1 places on $B \upharpoonright l_2$. We require d_2 to be large enough so that m_0 can copy all the wagers that φ_0 may place along strings of length l_2 beneath d_2 , and m_1 can copy all the wagers that φ_1 may place along strings of length l_2 beneath d_2 . Therefore, by the same reasoning as the calculation of d_1 , we would like d_2 to be at least $2^{l_2} \cdot l_2$. We then define l_3 similarly, with $l_{3,0} = l_2 + d_2$, and $l_{3,1}, l_{3,2}, l_{3,3}$ and l_3 as in the argument of Section 4.1.

Now suppose that A changes between d_1 and d_2 at stage s' . That is, $A_{s'} \upharpoonright d_1 = A_{s'-1} \upharpoonright d_1$, but there is $m \in [d_1, d_2)$ with $A_{s'}(m) \neq A_{s'-1}(m)$. We cannot change B below l_1 , but we can change B above l_1 . We therefore choose a string τ of length l_2 which minimises the martingales φ_0 and φ_1 . As \mathcal{R}_0 has stronger priority than \mathcal{R}_1 , we first look to minimise φ_0 . If we can, we change B to minimise φ_0 , and if we cannot, we then look to minimise φ_1 . If we can, we change B to minimise φ_1 , and if we cannot, we do not change B . In any case, we then choose some string ρ of length l_3 extending the current version of $B \upharpoonright l_2$ which has not been visited before, and let $B_{s'+1} = \rho 1^\omega$.

4.3.3. *Construction.* For every $e \in \omega$ and at every stage s we have a restraint $r_{e,s}$. During the construction we will say that “ m_e has copied φ_e 's wager on σ ” for some e and string σ . Let $d_{e,s}$ denote the length of longest string for which we have defined m_e by the beginning of stage s . Set $l_0 = 0$, $d_0 = 0$ and $l_1 = 5$. Given l_i , we set $d_i = 2^{l_i} \cdot l_i$,

and then given d_i , we set $l_{i+1,0} = l_i + d_i$ and for $0 \leq j \leq i$,

$$l_{i+1,j+1} = l_{i+1,j} + \max_{k \leq i} n(2^{l_{i+1,j}+1} m_k(\lambda)),$$

and let $l_{i+1} = l_{i+1,i+1} + 1$. Set $\gamma(l_i) = d_i$ and $\delta(d_i) = l_{i+1}$ for all i . At stage 0, we set $B_1 = 1^\omega$, $m_e(\lambda) = \varphi_e(\lambda)$ for all e , and $r_{e,1} = l_e$ for all e . At stage $s > 0$ do the following:

Case 1: there is $l \geq r_{e,s}$ such that $\varphi_e(B_s \upharpoonright (l+1)) > \varphi_e(B_s \upharpoonright l)$, m_e has copied φ_e 's wagers on $B_s \upharpoonright 1, \dots, B_s \upharpoonright l$, and m_e has not already copied φ_e 's wager on $B_s \upharpoonright (l+1)$. Let e be the least applicable, and l the least applicable for this e . Define m_e to wager $\varphi_e(B_s \upharpoonright (l+1)) - \varphi_e(B_s \upharpoonright l)$ on $A_s \upharpoonright (d_{e,s} + 1)$ and wager 0 on all other strings of length $d_{e,s} + 1$. We say that m_e has copied φ_e 's wager on $B_s \upharpoonright (l+1)$. Let i be such that $l \in (l_i, l_{i+1}]$. For all $e' > e$ with $r_{e',s} \leq l_{i+1}$, let $r_{e',s+1} = l_{i+2}$. Proceed to the next stage.

Case 2: $A_s(m) \neq A_{s+1}(m)$. Let m be the least applicable, and let i be such that $m \in (d_i, d_{i+1}]$. Choose the least e with $r_{e,s} < l_{i+1}$ such that there is τ of length l_{i+1} extending $B_s \upharpoonright l_i$ with $\max_{j \leq l_{i+1}} \varphi_{e,s+1}(\tau \upharpoonright j) < \max_{j \leq l_{i+1}} \varphi_{e,s+1}(B_s \upharpoonright j)$. For this e , choose an applicable string τ with $\min_\mu d(\tau, \mu)$ as large as possible, where the minimum is taken over all forbidden strings of length l_{i+1} extending $B_s \upharpoonright l_i$. If there is no such e , then let $\tau = B_s \upharpoonright l_{i+1}$. Choose a string ρ of length l_{i+2} extending τ such that $\rho \neq B_t$ for all $t \leq s$. Let $B_{s+1} = \rho 1^\omega$. Proceed to the next stage.

If neither case applies, proceed to the next stage.

4.3.4. Verification of the construction. First, we observe that $A \equiv_{\text{wt}} B$. Indeed, the uses $\gamma(l_i)$ are clearly computable, and so we have $\Gamma^A = B$ via the weak truth-table functional Γ . The consistency of Δ is a consequence of B never extending a forbidden string. Again the uses $\delta(d_i)$ are computable and so $\Delta^B = A$ via the weak truth-table functional Δ .

It remains to show that for all $e \in \omega$, \mathcal{R}_e is satisfied. Suppose by induction that all requirements of stronger priority than \mathcal{R}_e do not act after stage s^* . We show that if φ_e succeeds on B , then m_e succeeds on A , which is a contradiction to A being integer-valued random.

Let i_0 be least such that $l_{i_0} \geq r_{e,s^*}$. By the restraints imposed, we cannot change B below l_{i_0} for the sake of \mathcal{R}_e . Now suppose that at stage $s \geq s^*$ we see $\varphi_{e,s}(B_s \upharpoonright l) > \varphi_{e,s}(B_s \upharpoonright l_{i_0})$. Let l be the least applicable, and suppose i is such that $l \in (l_i, l_{i+1}]$. From stage s we have m_e copy φ_e 's wagers, and at some stage $s' \geq s$ we define m_e to wager $\varphi_e(B_{s'} \upharpoonright l) - \varphi_e(B_{s'} \upharpoonright (l-1))$ on $A_{s'} \upharpoonright d_{e,s'} + 1$. If m_e copies all of the wagers that φ_e makes on strings of length less than or equal to $l-1$, then m_e is defined on strings of length at most $2^{l-1} \cdot (l-1)$. Therefore $d_{e,s'} \leq 2^{l-1} \cdot (l-1) < d_{i+1}$.

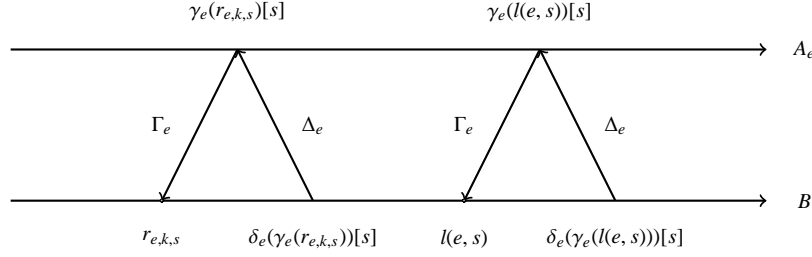
Suppose that at stage $t > s'$ we see A change below d_{i+1} . Then we will choose a string τ of length l_{i+1} extending $B_t \upharpoonright l_i$ with $\max_{j \leq l_{i+1}} \varphi_{e,t}(\tau \upharpoonright j) < \varphi_{e,t}(B_s \upharpoonright l)$. Taking the contrapositive, we see that if φ_e makes capital on B past l_{i_0} , then m_e makes capital on A . Therefore if φ_e succeeds on B then m_e succeeds on A .

5. COMPUTABLY ENUMERABLE DEGREES NOT CONTAINING INTEGER-VALUED RANDOMS

It is hardly surprising that there are c.e. degrees which do not contain integer-valued randoms. After all, computable enumerability hinders randomness, and indeed with respect to a sufficient level of randomness, c.e. sets are not random. However integer-valued randomness is sufficiently weak so that it has interesting interactions with computable enumerability. In this section we look at the question of which c.e. degrees do not contain integer-valued randoms. The first example of such degrees was given in Section 3.2 where we showed Theorem 1.7. Perhaps more surprising is the fact that there are c.e. degrees which are not array computable (so, by Corollary 1.5 they compute an integer-valued random) yet they do not contain integer-valued randoms. We prove this in Section 5.1, and extend it to a much stronger result (namely Theorem 1.8) in Section 5.2.

5.1. C.e. array noncomputable degrees not containing integer-valued randoms. We wish to construct an array noncomputable c.e. degree not containing an integer-valued random set. Let $(\Gamma_e, \Delta_e)_{e \in \omega}$ be an effective listing of all pairs of Turing functionals, and let $\langle D_n \rangle$ be the very strong array with $D_0 = \{0\}$, $D_1 = \{1, 2\}$, $D_2 = \{3, 4, 5\}$, ... We build a c.e. set B to satisfy the requirements

$$\begin{aligned} R_e: & (\exists n)(W_e \cap D_n = B \cap D_n) \\ N_e: & \Delta_e^B = A_e \wedge \Gamma_e^{A_e} = B \implies A_e \text{ is not integer-valued random.} \end{aligned}$$

Figure 3. A visualization of the reductions in requirement N_e

We build for each $e \in \omega$ an integer-valued martingale m_e , and replace the requirement N_e with the following requirements $N_{e,k}$ for all $k \geq 2$:

$$N_{e,k}: \Delta_e^B = A_e \wedge \Gamma_e^{A_e} = B \implies m_e \text{ wins at least } k \text{ dollars on } A_e.$$

We effectively order the requirements, making sure that if $k < k'$, then $N_{e,k}$ has stronger priority than $N_{e,k'}$. We say that R_e requires attention at stage s if

- (1) R_e has no follower at stage s , or
- (2) R_e has follower i at stage s and $W_{e,s} \cap D_i \neq B_s \cap D_i$.

For any $e \in \omega$, let $d_{e,s}$ be the length of the longest string σ for which $m_e(\sigma)$ is defined by stage s . We have for every $e \in \omega$ a restraint r_e . Let $r_{e,k,s} = \max D_i$ where i is the follower at stage s of any R -requirement of stronger priority than $N_{e,k}$. Let $l(e, s)$ be the length of agreement between $\Gamma_e^{A_e}$ and B at stage s ,

$$l(e, s) = \max\{x \mid (\forall y < x)(\Gamma_e^{A_e}(y)[s] = B(y)[s])\}.$$

We say that $N_{e,k}$ requires attention at stage s if

- (1) $m_e(A_{e,s} \upharpoonright d_{e,s}) = k - 1$,
- (2) $l(e, s) > r_{e,k,s}$,
- (3) $l(e, s) > \delta_e(\gamma_e(r_{e,k,s}))[s]$
- (4) $\gamma_e(l(e, s))[s] > d_{e,s}$

Figure 3 illustrates the reductions involved in requirement N_e .

Construction. At stage 0, let $B_0 = \emptyset$. Let $m_e(\lambda) = 1$ for all $e \in \omega$. At stage s , $s \geq 1$, find the requirement of strongest priority which requires attention at stage s .

Case 1: this is R_e . If R_e has follower i , enumerate $W_{e,s} \cap D_i$ into B . If R_e does not have a follower, appoint a fresh large follower for R_e .

Case 2: this is $N_{e,k}$. Let $\tau = A_e[s] \upharpoonright \gamma_e(l(e, s))[s]$. Define m_e to wager \$1 on τ , and bet neutrally on all other strings with length in $(d_{e,s}, |\tau|]$. Remove the followers of R -requirements of weaker priority than $N_{e,k}$.

Verification. It remains to show that each requirement requires attention only finitely often, and is met. Assume by induction that all requirements of stronger priority than R_e do not require attention after stage s . If R_e does not have a follower at stage s then it will be appointed one. This follower cannot be cancelled as requirements of stronger priority can no longer act. Suppose that R_e has follower i at stage s . If we ever see that $B_t \cap D_i \neq W_{e,t} \cap D_i$ then we will enumerate $W_{e,t} \cap D_i$ into B . As $|D_i| = i + 1$, R_e can require attention at most $i + 1$ many times after stage s . Then we will have $B \cap D_i = W_e \cap D_i$ and R_e is satisfied.

We claim that for all e , if $\Delta_e^B = A_e$ and $\Gamma_e^{A_e} = B$, then m_e is nondecreasing along A_e . Suppose we have $m_e(\sigma) = k - 1$ for some string σ . Suppose $N_{e,k}$ requires attention at stage t and we define $m_e(\tau) = k$. We remove the followers for R -requirements of weaker priority, and so only requirements of stronger priority than $N_{e,k}$ can

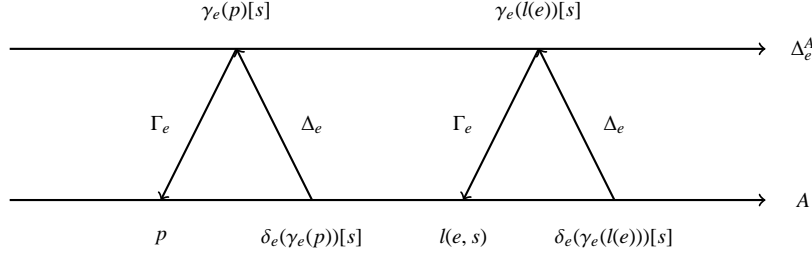


Figure 4. Diagram with the uses and where we bet, for the proof of Theorem 1.8.

enumerate elements into B that are below $\delta_e(\gamma_e(l(e, t)))[t]$. Suppose R_j has stronger priority than $N_{e,k}$. Then R_j can either enumerate elements below $r_{e,k,t}$, or if it is later injured, enumerate elements larger than $\delta_e(\gamma_e(l(e, t)))[t]$ into B . Suppose that R_j enumerates an element below $r_{e,k,t}$ into B . If Γ_e and Δ_e later recover at stage t' , $\tau' := A_e[t'] \upharpoonright \gamma_e(r_{e,k,t})[t]$ must be incomparable with $\tau = A_e[t] \upharpoonright \gamma_e(r_{e,k,t})[t]$; otherwise Δ_e will not have recorded the B -change and we could not have $\Delta_e^B = A_e$ and $\Gamma_e^{A_e} = B$. In particular, τ' must not extend either τ or its sibling (recall the definition of the *sibling* of a string, just after Definition 1.1). As m_e is defined to bet neutrally on strings of length $|\tau|$ that are not either τ or its sibling, we will have $m_e(A_e[t'] \upharpoonright |\tau|) \geq k - 1$. By induction, this holds for all k . This establishes the claim.

Now assume by induction that all requirements of stronger priority than $N_{e,k}$ do not require attention after some stage s . As $N_{e,j}$ for any $j < k$ does not require attention, we must have $m_e(A_{e,s} \upharpoonright d_{e,s}) = k - 1$. If $N_{e,k}$ does not require attention at any stage $t > s$ then the hypothesis of the requirement does not hold. Therefore $N_{e,k}$ is satisfied vacuously. If $N_{e,k}$ does require attention at stage $t > s$ then we define $m_e(\tau) = k$ for $\tau = A_e[t] \upharpoonright \gamma_e(l(e, t)))[t]$. R -requirements of stronger priority have finished acting, and so no numbers less than $r_{e,k,t}$ enter B after stage t . We remove the followers for R -requirements of weaker priority. When they are appointed new followers they will choose fresh numbers, and so all enumerations into B after stage t will be larger than $\delta_e(\gamma_e(l(e, t)))[t]$. As B cannot change below $\delta_e(\gamma_e(l(e, t)))[t]$, $A_e = \Delta_e^B$ cannot change below $\gamma_e(l(e, t)))[t]$. Therefore $\tau < A_e$ and $m_e(A_e \upharpoonright \gamma_e(l(e, t)))[t]) = k$. By the previous claim, $m_e(A_e) \geq k$, and $N_{e,k}$ is satisfied.

5.2. A high_2 c.e. degree not containing integer-valued randoms. In this section we prove Theorem 1.8. Nies, Stephan and Terwijn showed that every high degree contains a computably random set, and so a fortiori, an integer-valued random set. It is instructive though to see why we cannot build a high degree that does not contain an integer-valued random set. This will give us some insight as to why the construction works when we only require our set to be high_2 . So that the degree of A does not contain an integer-valued random set, we meet the requirements

$$\mathcal{N}_e: \Gamma_e^{\Delta_e^A} = A \text{ total} \implies \Delta_e^A \text{ is not integer-valued random}$$

where (Γ_e, Δ_e) is an enumeration of pairs of Turing functionals. We break the requirement \mathcal{N}_e into the following subrequirements $\mathcal{N}_{e,k}$.

$$\mathcal{N}_{e,k}: \Gamma_e^{\Delta_e^A} = A \text{ total} \implies m_e \text{ wins at least } k \text{ dollars on } \Delta_e^A$$

where m_e is an integer-valued martingale we build for the sake of \mathcal{N}_e . Suppose that the martingale m_e has won $\$k - 1$ on $\Delta_e^A \upharpoonright n$. The basic strategy to win another dollar is to first pick a large location marker p . Let $l(e)[s]$ be the length of agreement between A and $\Gamma_e^{\Delta_e^A}$ at stage s . If we later see that $l(e)[s] > p$ and $l(e)[s] > \delta_e(\gamma_e(p))[s]$, then we define m_e to wager $\$1$ on $\Delta_e^A[s] \upharpoonright \gamma_e(l(e))[s]$ and bet neutrally elsewhere, and freeze A below $\delta_e(\gamma_e(l(e)))[s]$. If we are successful in freezing A , then m_e wins $\$1$ on Δ_e^A . If A changes below p at stage $s' > s$, then if we are to have $\Gamma_e^{\Delta_e^A} = A$, $\Delta_e^A[s'] \upharpoonright \gamma_e(p)[s]$ is incomparable with $\Delta_e^A[s] \upharpoonright \gamma_e(p)[s]$ and m_e does not lose any capital along Δ_e^A . We can then try the basic strategy again.

To make A high we would define a functional Λ such that $\lim_k \Lambda^A(x, k) = \text{Tot}(x)$, where Tot is the canonical Π_2^0 -complete set. The basic strategy for the highness requirement is to define $\Lambda^A(x, s) = 0$ for larger and larger s with some big use $\lambda(x, s)$. When we see $\varphi_x(s') \downarrow$ for all $s' \leq s$, then for each $s' \leq s$ we enumerate the current use $\lambda(x, s')$ into A (if currently $\Lambda^A(x, s') = 0$) and redefine $\Lambda^A(x, s') = 1$ with use -1 , i.e. the axiom defining $\Lambda^A(x, s') = 0$ does not depend on A . This strategy will succeed as long as we are prevented from redefining $\Lambda^A(x, s)$ from 0 to 1 at most finitely often.

Let us see how these strategies might interact. Suppose at stage s we saw the $l(e)$ computations converge and defined more of m_e . The highness requirement, if unrestrained, can destroy the $l(e)$ computations and cause m_e to lose capital if it enumerates a marker between p and $\delta_e(\gamma_e(l(e)))[s]$. Therefore if m_e is to ever win money along Δ_e^A we must impose restraint on A . The problem is that the strategies for \mathcal{N}_e may gang up and impose restraint on *all* markers $\lambda(x, s)$; every time a marker is defined we may define m_e and impose restraint, and never allow the marker to be enumerated. If $\text{Tot}(x) = 1$ we will never be able to correct $\Lambda^A(x, s)$ to be 1 and the limit will be incorrect.

Another approach we might take would be to capriciously enumerate the markers which occur below $\delta_e(\gamma_e(l(e)))[s]$. If we do this and always have some marker below the use, we will be able to conclude that $\Gamma_e^{\Delta_e^A} \neq A$ (this argument is given in full in the verification below). However, the problem now is that we might not have wanted to enumerate the markers. If $\text{Tot}(x) = 0$ and we capriciously enumerate all markers $\lambda(x, s)$ and redefine $\Lambda^A(x, s)$ to be 1, the limit will be incorrect.

To make A high₂, we need to instead define a functional Λ such that

$$\lim_m \lim_t \Lambda^A(x, m, t) = \text{Cof}(x)$$

where $\text{Cof} = \{x \mid W_x \text{ is cofinite}\}$ is the canonical Σ_3^0 -complete set. The double limit means that we may be wrong on a finite number of the m while still satisfying the requirement. This will allow us to employ the capricious enumeration strategy successfully. We have the requirements

$$\mathcal{H}_x: \lim_m \lim_t \Lambda^A(x, m, t) = \text{Cof}(x)$$

as well as the \mathcal{N}_e from above. The construction will use a priority tree. For each global requirement \mathcal{N}_e we have several nodes devoted to meeting \mathcal{N}_e , each equipped with a guess as to the outcomes of stronger priority requirements. Such a node will be called a *mother node* devoted to \mathcal{N}_e . Each such node τ builds its own martingale m_τ . We argue in the verification that for every $e \in \omega$ there is some node τ such that m_τ succeeds on Δ_e^A . For each subrequirement $\mathcal{N}_{e,k}$ we have several nodes devoted to meeting $\mathcal{N}_{e,k}$. Such a node will be called a *worker node* devoted to $\mathcal{N}_{e,k}$, and will occur below a mother node τ devoted to \mathcal{N}_e . For the longest such τ , we say that σ 's mother node is τ . When we reach a node σ devoted to $\mathcal{N}_{e,k}$, we choose a fresh location marker p for σ , and place a *link* from σ back up to τ . The link can be seen as testing the hypothesis of \mathcal{N}_e . The length of agreement between A and $\Gamma_e^{\Delta_e^A}$ will be measured at τ . When we next arrive at τ and see the length of agreement computations converge, this further confirms the hypothesis of \mathcal{N}_e . We travel the link to σ , define more of the martingale m_τ , and then remove the link.

The requirement \mathcal{H}_x will have nodes $\beta_{x,m}$ for $m \in \omega$. The node $\beta_{x,m}$ tests whether $[m, \infty) \subseteq W_x$. Note that this is a Π_2 test. Such a node will have outcomes ∞ , which corresponds to the Π_2 test infinitely often looking correct, and f , corresponding to the finite outcome. The $\beta_{x,m}$ nodes will be responsible for defining $\Lambda^A(x, m, t)$ for each t . As we do not know the true path in advance, each path through the priority tree will have a $\beta_{x,m}$ node, which collectively will define $\Lambda^A(x, m, t)$ for all t .

The basic strategy for $\beta_{x,m}$ is to define $\Lambda^A(x, m, t) = 0$ for larger and larger t with some big use $\lambda(x, m, t)$. If we see $[m, s] \subseteq W_x$ then for each $s' \leq s$ we enumerate the current use $\lambda(x, m, s')$ into A (if currently $\Lambda^A(x, m, s') = 0$) and redefine $\Lambda^A(x, m, s') = 1$ with use -1 , i.e. the axiom defining $\Lambda^A(x, m, s') = 0$ does not depend on A . It is important to note that the markers $\lambda(x, m, s)$ are *shared* by all the $\beta_{x,m}$ nodes. Whether we succeed in enumerating the markers and updating Λ as needed will depend on how the construction proceeds.

We will now describe how these requirements interact and what modifications we need to make to the priority tree as a consequence. First we consider the situation where we have \mathcal{H}_x of lower priority than \mathcal{N}_e (which is

associated with the mother node τ). The problem is the following. Suppose we have a situation with nodes $\tau < \beta_{x,m} < \sigma$ where σ is devoted to $N_{e,k}$ and τ is σ 's mother node. That is, while σ has higher *global* priority than $\beta_{x,m}$, its *local* priority is lower. Suppose at some stage σ picks a location marker $p(\sigma)$ and creates a link back to τ at stage s_0 . At a later stage s_1 we get to τ , see the necessary computations converge, and would like to travel the link and define the martingale. This causes no problem if $\sigma \geq \beta_{x,m} \hat{f}$, but there are problems if $\sigma \geq \beta_{x,m} \hat{\infty}$. We will require the computations $l(\tau)[s]$ to be τ -correct; that is, all guesses τ makes about the enumeration of markers below $\delta_e(\gamma_e(l(e)))[s]$ have already occurred.

The trouble is that at stage s_1 there now might be some marker $\lambda = \lambda(x, m, q)$ which is greater than $p(\sigma)$, but below the use of $l(\tau)[s_1]$. We may not yet want to enumerate λ into A because the Σ_2 outcome may now be looking correct at $\beta_{x,m}$ (that is, we might think $\text{Cof}(x) = 0$). If we did define the martingale, since $\beta_{x,m}$ has higher priority than σ , any restraint imposed at s_1 may not be successful since $\beta_{x,m}$ might later put λ into A . This could potentially cause our martingale m_τ to lose all its capital, and it could never bet again.

The solution to this problem is as follows. When we hit τ , if there is some link to a node σ and there is some λ as above, we immediately enumerate any λ below the use of $l(\tau)[s]$ into A , but we do *not* define the martingale. This means that $\beta_{x,m}$ cannot later use λ to make m_τ lose capital. If there is no such λ then we do define the martingale, since we can be sure that σ is satisfied provided that it is on the true path. This is the situation we would like, but failing that, we would like to get a global win on N_e . In the case that such a λ exists, we travel the link from τ to σ , enumerate all applicable markers, but we do *not* delete the link. Because of this, we need to add a new outcome to σ . Therefore σ will have outcomes g and d . The outcome g will be played when we perform the capricious enumeration of λ as above. The outcome d will be played when we define the martingale. Suppose we have some worker node σ with location marker $p(\sigma)$ which always has some λ below the use of $l(\tau)[s]$. We will then define $\Lambda^A(x, m, t)$ to have limit 1 for any pair (x, m) such that $\tau < \beta_{x,m} < \sigma$; note that there are only finitely many pairs (x, m) . We will also have a link from τ to σ for almost all stages. This corresponds, however, to a global win on N_e , since $p(\sigma)$ is a witness to the fact that $\delta_e(\gamma_e(l(e)))$ does not exist and so $\Gamma_e^{\Delta_e} \neq A$. The permanent link may cause us to skip over other mother nodes, which would mean we cannot meet their requirements. We therefore *restart* all N -requirements of weaker priority than N_e under the g outcome of σ . We do this by assigning the requirements $N_{e'}$ for $e' > e$, as well as their subrequirements $N_{e',k}$ for $k \in \omega$, to nodes below $\sigma \hat{g}$ in some fair way. We do not restart any β nodes, since $\Lambda^A(x, m, t)$ will be defined to be 1 for the finitely many x and m with $\tau < \beta_{x,m} < \sigma$. This will mean that for a finite number of m , $\lim_t \Lambda^A(x, m, t)$ may be incorrectly outputting 1 instead of 0. This is fine though, since we will only lose on a finite number of the m and still can satisfy \mathcal{H}_x . The sacrifice of losing on an m will only be made when we can ensure a global win on a τ node of stronger priority.

We now come to the situation where we have an \mathcal{H}_x of *higher* global priority than the N_e associated with τ . We now cannot allow τ to capriciously enumerate all the markers belonging to $\beta_{x,m}$ if f is $\beta_{x,m}$'s true outcome. We now describe our solution to this problem.

First suppose that $\beta_{x,m}$ and $\beta_{x,n}$ are worker nodes devoted to \mathcal{H}_x with $m < n$. If $\beta_{x,n}$ occurs below $\beta_{x,m} \hat{\infty}$, then $\beta_{x,n}$ is guessing that $[m, \infty) \subseteq W_x$. Therefore $\beta_{x,n}$ must also be guessing that $[n, \infty) \subseteq W_x$ as $n > m$, and so $\beta_{x,n}$ will have only the ∞ outcome.

Suppose that τ is below the f outcome of any $\beta_{x,m}$ node with $m < n$. We will restart τ below $\beta_{x,n} \hat{\infty}$. Consider the situation $\beta_{x,n} \hat{\infty} \leq \tau < \beta_{x,n'} < \sigma$ with $n' > n$. If there is a link from τ to σ then capriciously enumerating the markers $\lambda(x, n', t)$ into A will not injure $\beta_{x,n'}$ since this is what $\beta_{x,n'}$ would like to do anyway. Therefore \mathcal{H}_x cannot be injured in this situation. We show in Lemma 5.1 that such a τ can be restarted only finitely many times.

There is one last problem. Suppose we have $\tau < \beta_{x,m} < \sigma$ with a link (τ, σ) . As σ 's mother is above $\beta_{x,m}$, we must have $\beta_{x,m} \hat{f} \leq \sigma$. If the link is permanent then $\beta_{x,m}$ will not be able to enumerate its markers. As \mathcal{H}_x has higher global priority than N_e , this is not a situation we want. We employ the following technique from Downey-Stob [DS97]. When we hit τ , we realize that if there is a link from τ down then this may be a potentially permanent link. We first perform a *scouting report* to see where we would go if there were no link around. If we were to go to a node γ to the left of σ then we will erase the link and actually go to γ instead. This ensures that if $\beta_{x,m} \hat{\infty}$ is $\beta_{x,m}$'s true outcome, then it will be able to enumerate its markers.

The Priority Tree. Our priority tree, PT, will have three types of nodes. The first type are *mother* nodes τ , which have outcomes ∞ and f , and will be assigned to some global requirement N_e . We write $e(\tau) = e$. The next type are *worker* nodes σ which are devoted to a subrequirement of some N_e , and hence will be assigned some e, k . We write $e(\sigma) = e, k(\sigma) = k$. We form the tree so that such σ occur below some τ with $e(\tau) = e$. For the longest such τ with $e(\tau) = e$, we will write $\tau(\sigma) = \tau$. This is to indicate that τ is σ 's mother. σ has outcomes g and d with $g < d$. Finally we have nodes β which are devoted to some \mathcal{H}_x , and hence will be assigned some x, m . We write $x(\beta) = x, m(\beta) = m$, or simply $\beta_{x,m}$. β will have outcomes ∞ and f , with $\infty < f$, unless β occurs below $\beta' \wedge \infty$ for some β' with $x(\beta') = x(\beta)$, in which case β has only the single outcome ∞ .

We now assign requirements and subrequirements to nodes on the tree. In a basic infinite injury argument we would have all nodes of the same level working for the same requirement. However in our case, as we must restart τ nodes, it is more complicated. We use *lists* of, for example, Soare [Soa87, Chapter XIV] for this. We will have three lists, L_0, L_1 , and L_2 , which keep track of indices for τ, σ and β nodes, respectively.

$n = 0$. Let λ be devoted to N_0 , and let $L_0(\lambda) = L_1(\lambda) = L_2(\lambda) = \omega$.

For $n > 0$, let $\gamma \in PT$ be of the form $\delta \wedge a$. Adopt the first case below to pertain, letting $L_i(\gamma) = L_i(\delta)$ unless otherwise mentioned.

Case 1. δ is devoted to N_e .

Case 1a. $a = f$. $L_0(\gamma) = (L_0(\delta) - \{e\}) \cup \{e' \mid e' > e\}$

$$L_1(\gamma) = (L_1(\delta) - \{\langle e, k \rangle \mid k \in \omega\}) \cup \{\langle e', k \rangle \mid e' > e, k \in \omega\}.$$

Case 1b. $a = \infty$. $L_0(\gamma) = L_0(\delta) - \{e\}$.

Case 2. δ is devoted to $N_{e,k}$.

Case 2a. $a = g$. Define the lists as in Case 1a.

Case 2b. $a = d$. Let $L_1(\gamma) = L_1(\delta) - \{\langle e, k \rangle\}$.

Case 3. δ is devoted to \mathcal{H}_x with $m(\delta) = m$.

Case 3a. $a = f$. $L_2(\gamma) = L_2(\delta) - \{\langle x, m \rangle\}$.

Case 3b. $a = \infty$. Let $L_0(\gamma)$ be the union of $L_0(\delta)$ with

$$\{e \mid (\exists \tau)(\forall \beta)(e(\tau) = e \wedge x(\beta) = x \wedge x < e \wedge \beta < \tau < \delta \implies \beta \wedge f \leq \tau)\}$$

and let $L_1(\gamma)$ be the union of $L_0(\delta)$ with

$$\{\langle e, k \rangle \mid (\exists \tau)(\forall \beta)(e(\tau) = e \wedge x(\beta) = x \wedge x < e \wedge \beta < \tau < \delta \implies \beta \wedge f \leq \tau), k \in \omega\}.$$

Also let $L_2(\gamma) = L_2(\delta) - \{\langle x, m \rangle\}$.

Having defined the lists, we now assign requirements to nodes of the priority tree as follows. Let $\gamma \in PT$ and i be the least element of $L_0(\gamma) \cup L_1(\gamma) \cup L_2(\gamma)$. If $i \in L_0(\gamma)$, let γ be a mother node devoted to N_i . If $i \in L_1(\gamma) - L_0(\gamma)$ and $i = \langle e, k \rangle$, let γ be a worker node devoted to $N_{e,k}$. Otherwise $i = \langle x, m \rangle \in L_2(\gamma)$ and we let γ be worker node devoted to \mathcal{H}_x with $m(\gamma) = m$.

Lemma 5.1 (Finite injury along any path lemma). *For every path $h \in [PT]$ and every $e, k \in \omega$,*

- (1) $(\exists^{<\infty} \alpha < h)(e(\alpha) = e \wedge h(|\alpha|) = g)$,
- (2) $(\exists^{<\infty} \alpha < h)(\alpha \text{ devoted to } N_e)$,
- (3) $(\exists^{<\infty} \alpha < h)(\alpha \text{ devoted to } N_{e,k})$.

Proof. (1) and (3) are standard. For (2), a node τ devoted to N_e is restarted below $\beta \wedge \infty$ if $x(\beta) < e$ and τ has been below only the f outcomes of nodes devoted to \mathcal{H}_x . Once restarted, it can no longer be restarted below any other β' with $x(\beta') = x(\beta)$. Thus τ is restarted finitely many times. \square

The construction below will proceed in substages. We will append a subscript t to a parameter G , so that G_t denotes the value of G at substage t of the construction. As usual all parameters hold their value unless they are initialized. When initialization occurs they become undefined, or are set to zero as the case may be. We will append a parameter $[s]$, when necessary, to denote stage s . We may write (s, t) to denote substage t of stage s .

If we visit a node ν at stage (s, t) we will say that (s, t) is a genuine ν -stage. It might be that we do not visit ν at stage (s, t) , rather we visit some ν' extending ν . In this case we say that (s, t) is a ν -stage, and hence a ν -stage may not be genuine. In fact, should we put in place some permanent link (τ, σ) with $\tau < \nu < \sigma$, then ν might only ever be visited finitely often. However, this is when $\sigma^\wedge g$ is the true outcome for some higher priority τ , and we would claim that a new version of ν would live below outcome g of σ . We will eventually define the genuine true path as those nodes that are on the leftmost path visited infinitely often, and for which there are infinitely many genuine stages.

We have for each node on the priority tree γ and for all $x, m \in \omega$ a restraint $c_\gamma(x, m)$. These restraints will be initially set to zero in the construction, and will only be increased when γ is a mother node devoted to some N_e . We say that a computation $\Xi^A(x)[s]$ is τ -correct if $(\forall q \leq s)(\forall \beta)$ the condition

$$(\beta^\wedge \infty \leq \tau \wedge x(\beta) = x \wedge m(\beta) = m \wedge \max_{\tau' \leq \beta} c_{\tau'}(x, m)[s] < \lambda(x, m, q) < \xi^A(x)[s])$$

implies $\lambda(x, m, q) \in A[s]$. If τ is a mother node devoted to N_e , then let

$$l_1(\tau)[s] = \max\{x \mid (\forall y < x)(\Gamma_e^A(y)[s] = A[s](y)) \text{ via a } \tau\text{-correct computation}\}.$$

Let $m_1(\tau)[s] = \max\{l_1(\tau)[s'] \mid s' < s \text{ is a genuine } \tau\text{-stage}\}$. If τ is a node devoted to \mathcal{H}_x with $m(\tau) = m$, then let $l_2(\tau)[s] = \max\{y \mid [m, y] \subseteq W_x[s]\}$. Let

$$m_2(\tau)[s] = \max\{l_2(\tau)[s'] \mid s' < s \text{ is a genuine } \tau\text{-stage}\}.$$

For τ a mother node devoted to some N_e , let $d_\tau[s]$ denote the length of the longest string for which m_τ is defined by stage s .

Construction. At stage 0 set $\Lambda^A(0, 0, 0) = 0$ with some large use $\lambda(0, 0, 0)$. Set $c_\gamma(x, m)[0] = 0$ for all $x, m \in \omega$ and all nodes γ on the priority tree. Set $m_\tau(\lambda) = 1$ for all nodes τ devoted to some requirement N_e . Stage $s + 1$ will proceed in substages $t \leq s$. As usual, we will generate a set of accessible nodes, $\text{TP}[s + 1]_t$, and will automatically initialize nodes α to right of $\text{TP}[s + 1]_t$. A node is initialized by removing its location marker and removing any link to or from the node.

Substage 0. Define $\text{TP}[s + 1]_0 = \lambda$, the empty string. Let $\Lambda^A(x, m, s + 1) = 0$ with some large use $\lambda(x, m, s + 1)$ for all $x, m \leq s + 1$.

Substage $t + 1 \leq s + 1$. We will be given a string $\gamma = \text{TP}[s + 1]_t$. Adopt the first case to pertain below.

Case 1. γ is a mother node devoted to N_e .

Subcase 1a. There is a link (γ, σ) for some node σ . We perform the *scouting report* by computing the string γ' that would be $\text{TP}[s + 1]$ were there no link. If $\gamma' <_L \sigma$, remove the link (γ, σ) , let $\text{TP}[s + 1]_{t+1} = \gamma' \upharpoonright (|\gamma| + 1)$, and go to substage $t + 2$. If $\gamma' \not<_L \sigma$, see whether $l_1(\gamma)[s + 1] > p(\sigma)$, $\delta_e(\gamma_e(p(\sigma)))[s + 1]$ with $\gamma_e(p(\sigma))[s + 1] > d_\gamma[s + 1]$.

Subcase 1a.1. No. Set $\text{TP}[s + 1]_{t+1} = \tau^\wedge f$ and go to substage $t + 2$.

Subcase 1a.2. Yes and for some node β devoted to \mathcal{H}_x with $m(\beta) = m$ and $\gamma < \beta^\wedge \infty \leq \sigma$, there is a marker $\lambda(x, m, q)$ with $p(\sigma) \leq \lambda(x, m, q) \leq \delta_e(\gamma_e(l_1(\gamma)))[s + 1]$. Our action is to set $\text{TP}[s + 1]_{t+1} = \sigma$. We refer to this action as traveling the link. Go to substage $t + 2$.

Subcase 1a.3. Otherwise, set $\text{TP}[s + 1]_{t+1} = \sigma$ and go to substage $t + 2$.

Subcase 1b. There is no link from γ . Let $\text{TP}[s + 1]_{t+1} = \gamma^\wedge \infty$.

Case 2. γ is a worker node devoted to $N_{e,k}$.

Subcase 2a. We were in subcase 1a.2 in the previous substage. Enumerate all markers as in the previous substage into A . Let $\text{TP}[s + 1]_{t+1} = \gamma^\wedge g$.

Subcase 2b. We were in subcase 1a.3 in the previous substage. Define $m_{\tau(\gamma)}$ to wager \$1 on $\Delta_e^A[s + 1] \upharpoonright \gamma_e(l_1(\tau(\gamma)))[s + 1]$ and bet neutrally on all other strings up to and including that length. For all x, m such that there is $\beta > \gamma$ with $x(\beta) = x, m(\beta) = m$, let $c_{\tau(\gamma)}(x, m)[s + 1] = \delta_e(\gamma_e(l_1(\tau(\gamma)))[s + 1]$. Remove the link $(\tau(\gamma), \gamma)$. Let $\text{TP}[s + 1]_{t+1} = \gamma^\wedge d$.

Subcase 2c. We did not travel a link to arrive at γ . If $m_{\tau(\gamma)}(A_s \upharpoonright d_{\tau(\gamma)}[s+1]) \geq k$, let $\text{TP}[s+1]_{t+1} = \gamma \hat{d}$, and go to substage $t+2$. If not, choose a fresh large follower $p(\gamma)$ for γ , place a link $(\tau(\gamma), \gamma)$, and go to stage $s+2$.

Case 3. γ is a node devoted to \mathcal{H}_x with $m(\gamma) = m$. Consider the immediate successors of γ on the priority tree.

Case 3a. The immediate successors of γ are $\gamma \hat{\infty}$ and $\gamma \hat{f}$. See whether $l_2(\gamma)[s+1] > m_2(\gamma)[s+1]$.

Case 3a.1. Yes. For all $q \leq m_2(\gamma)[s+1]$, if $\lambda(x, m, q) > \max_{\tau \leq \gamma} c_\tau(x, m)[s+1]$ and $\Lambda^A(x, m, q) = 0$, enumerate $\lambda(x, m, q)$ into A and define $\Lambda^A(x, m, q) = 1$ with use -1 . Set $\text{TP}[s+1]_{t+1} = \gamma \hat{\infty}$.

Case 3a.2. No. Set $\text{TP}[s+1]_{t+1} = \gamma \hat{f}$.

Case 3b. The immediate successor of γ is $\gamma \hat{\infty}$. For all $q \leq m_2(\gamma)[s+1]$, if

$$\lambda(x, m, q) > \max_{\tau \leq \gamma} c_\tau(x, m)[s+1] \text{ and } \Lambda^A(x, m, q) = 0$$

enumerate $\lambda(x, m, q)$ into A and define $\Lambda^A(x, m, q) = 1$ with use -1 . Set $\text{TP}[s+1]_{t+1} = \beta \hat{\infty}$.

Verification. We define TP, the *true path*, to be the leftmost path visited infinitely often. This clearly exists as the priority tree is finite-branching. We define GTP, the *genuine true path*, to be those $\alpha < \text{TP}$ such that there are infinitely many *genuine* α -stages.

Lemma 5.2. *For every $e \in \omega$ there is a node τ devoted to N_e on GTP.*

Proof. Let τ be the longest node devoted to N_e on TP, which exists by the finite injury along any path lemma. We claim that τ is on GTP. Suppose otherwise. Then it must be the case that there are τ' and σ on GTP such that $\tau' < \tau < \sigma$ and the link (τ', σ) is there at almost all stages. This implies that $\sigma \hat{g}$ is on GTP. On the priority tree, if $\tau_1 < \tau_2$ then $e(\tau_1) < e(\tau_2)$. Now as σ links to its mother node τ' and $\tau' < \tau$ we must have $e(\tau') < e(\tau) = e$. But then by the construction of the priority tree, there is an N_e node below $\sigma \hat{g}$, contradicting the hypothesis that τ is the longest such. \square

Lemma 5.3. *For every $e \in \omega$, N_e is satisfied.*

Proof. Let τ be the longest node on GTP devoted to N_e . First suppose that $\tau \hat{f} < \text{GTP}$. Then $\Gamma_e^{\Delta_e^A} \neq A$ and N_e is vacuously satisfied. If there is a permanent link (τ, σ) for some node σ , then we claim that $\Gamma_e^{\Delta_e^A} \neq A$. For contradiction suppose $\Gamma_e^{\Delta_e^A} = A$, and suppose the link was placed at stage s_0 . Then there is a stage $s > s_0$ and uses $\delta_e(\gamma_e(l_1(\tau))) = a_1, \gamma_e(l_1(\tau)) = a_2$, such that $\Gamma_e^{\Delta_e^A \upharpoonright a_1 \upharpoonright a_2}[s] \leq A \upharpoonright a_1$. However if this were the case, at the next genuine τ -stage greater than s we will see that these computations have converged, play the d outcome, and remove the link. This contradicts the fact that the link is permanent. This establishes the claim.

Now suppose there is no such permanent link. We will show that m_τ succeeds on Δ_e^A . We must ensure that m_τ 's capital does not decrease along Δ_e^A . Fix $k \in \omega$, and let $\sigma' > \tau$ be the first node devoted to $N_{e,k}$ that is visited. Suppose we visit σ' first at stage s_0 . At stage s_0 we assign σ' a fresh large location marker $p(\sigma')$ and link back to τ . Let s_1 be the stage at which we first define m_τ to win k on $\Delta_e^A[s_0]$. As we acted in case (2b) of the construction, we did not play the g outcome at stage s_1 and so there were no markers belonging to any nodes β such that $\tau < \beta \hat{\infty} \leq \sigma'$ below the use of our computations. The computations are τ -correct at stage s_1 and restraint is imposed on requirements of weaker priority than σ' . Therefore the only markers which can be enumerated below the use are those belonging to nodes β such that $\tau < \beta \hat{f} \leq \sigma'$. As $\beta \hat{f}$ was visited at stage s_0 , there is at least one marker, namely $\lambda(x, m, s_0)$ where $x = x(\beta)$ and $m = m(\beta)$, that has not been enumerated into A by stage s_0 . The location marker $p(\sigma')$ was chosen to be large at the substage when σ' was visited, and so is larger than $\lambda(x, m, s_0)$. If at some later stage t we enumerate $\lambda(x, m, s_0)$ into A , then A changes below $p(\sigma')$. If we are to have $\Gamma_e^{\Delta_e^A} = A$, then $\Delta_e^A[t] \upharpoonright \gamma_e(p(\sigma'))[s_0]$ is incomparable with $\Delta_e^A[s_0] \upharpoonright \gamma_e(p(\sigma'))[s_0]$. Therefore $m_\tau(\Delta_e^A[t] \upharpoonright d_\tau[t]) = k - 1$, and so m_τ has not decreased in capital along Δ_e^A . If we later arrive at β , we will play $\beta \hat{\infty}$ and initialize σ' as it is to the right of $\beta \hat{\infty}$. If we visit σ' again, a new location marker will be chosen, which must be larger than at least one marker of any node β such that $\tau < \beta \hat{f} \leq \sigma'$.

Let s_0 be the least genuine τ -stage. As τ is genuinely visited at stage s_0 there can be no link (τ', σ') at stage s_0 with $\tau' < \tau < \sigma'$. Fix $k \in \omega$ and suppose for contradiction that $\sigma < \text{TP}$ devoted to $\mathcal{N}_{e,k}$ is never genuinely visited. Then there is some permanent link (τ'', σ'') with $\tau < \tau'' < \sigma < \sigma''$. Suppose the link (τ'', σ'') was placed at stage s_1 . Then as σ'' is genuinely visited at stage s_1 and $\sigma < \sigma''$, σ is genuinely visited at stage s_1 . Contradiction. At stage s_1 we define the location marker $p(\sigma)$ and create a link (τ, σ) . Let s_2 be the stage at which we travel the link to σ and define m_τ to win $\$k$ on $\Delta_e^A[s_2]$. If we visit a node that is below τ but to the left of σ then, as in the previous paragraph, m_τ will then have capital $k - 1$. However $\sigma < \text{TP}$, and so this will occur only finitely many times. Let s_3 be the greatest stage at which σ is initialized. We remove any link over σ as part of the initialization. At the next σ -stage after s_3 we will genuinely visit σ and place a link if we see that $m_\tau(\Delta_e^A[s_3] \upharpoonright d_\tau[s_3]) < k$. Let s' be the stage at which we travel the link to σ and define more of m_τ . As we never visit a node to the left of σ , no marker belonging to a node β such that $\tau < \beta \wedge f \leq \sigma$ will be enumerated after stage s' . As in the previous paragraph, A cannot change below the use $\delta_e(\gamma_e(l(\tau)))[s']$ and $m_\tau(\Delta_e^A) \geq k$. \square

Lemma 5.4. *For every $x \in \omega$, \mathcal{H}_x is satisfied.*

Proof. Suppose $x \notin \text{Cof}$. Let $\beta < \text{TP}$ be the node devoted to \mathcal{H}_x with $m(\beta)$ least such that $\beta_{x,m}$ is not permanently linked over by τ 's of stronger priority for all $m \geq m(\beta)$. Let $m(\beta) = m_0$. We show that $\lim_t \Lambda^A(x, m, t) = 0$ for all $m \geq m_0$. We will have $\Lambda^A(x, m, t) = 0$ unless the marker $\lambda(x, m, t)$ is enumerated into A . Thus we must show that we eventually stop enumerating the markers $\lambda(x, m, t)$ into A . As $[m_0, \infty) \not\subseteq W_x$, the Π_2 test which measures whether $[m, \infty) \subseteq W_x$ will eventually always say “no”. So eventually the $\beta_{x,m}$ nodes will stop putting their markers into A and redefining $\Lambda^A(x, m, t)$. Therefore the only way we will enumerate the markers is if there is a link (τ, σ) over $\beta_{x,m}$ with $\tau < \beta_{x,m} \wedge \infty \leq \sigma$ and τ is accessible. We must show that if there is such a link then τ is accessible at only finitely many stages.

For τ with $e(\tau) > x$, we will restart τ below $\beta_{x,m} \wedge \infty$. If $\beta \wedge \infty \leq \tau < \beta_{x,m_1} \leq \sigma$ and a link (τ, σ) is placed over β_{x,m_1} for some $m_1 > m$, then as $\beta \wedge \infty$ can be visited at most finitely many times, only finitely many markers $\lambda(x, m_1, t)$ will be enumerated.

Now suppose τ is above β and we place a link (τ, σ) over β . Subcase (2a) of the construction will enumerate markers $\lambda(x, m, t)$ only if $\beta_{x,m} \wedge \infty \leq \sigma$. As σ must be below $\beta \wedge f$, the markers $\lambda(x, m, t)$ will not be enumerated, and we will have $\lim_t \Lambda^A(x, m, t) = 0$.

Finally, if $\beta'_{x,m}$ is another node on another path of the priority tree which is visited infinitely often, we must ensure that it does not enumerate all of the markers $\lambda(x, m, t)$. The Π_2 test performed at $\beta'_{x,m}$ is the same test which is performed at $\beta_{x,m}$ and so will eventually always say “no”. Therefore the marker $\lambda(x, m, t)$ will only be enumerated if there is a link (τ, σ) over $\beta'_{x,m}$ with $\beta'_{x,m} \wedge \infty \leq \sigma$. As $\beta'_{x,m}$ is to the right of TP it will be initialized infinitely many times. Any link over $\beta'_{x,m}$ will be removed as part of the initialization, and so $\beta'_{x,m}$ is not permanently linked over. The marker $\lambda(x, m, t)$ is enumerated capriciously only if $\beta'_{x,m} \wedge \infty \leq \sigma$. For the marker to be enumerated infinitely many times we must visit σ below $\beta'_{x,m} \wedge \infty$ and place a link back to τ . However if $\beta'_{x,m} \wedge \infty$ is visited infinitely many times, this contradicts $x \notin \text{Cof}$. Therefore $\beta'_{x,m}$ cannot enumerate infinitely many of its markers.

Now suppose that $x \in \text{Cof}$ and $[m_0, \infty) \subseteq W_x$. Let $\beta < \text{TP}$ be the node devoted to \mathcal{H}_x with $m(\beta)$ least such that $\beta_{x,m}$ is not permanently linked over by τ 's of stronger priority for all $m \geq m(\beta)$. Let $m' = \max\{m(\beta), m_0\}$. We show that $\lim_t \Lambda^A(x, m, t) = 1$ for all $m \geq m'$.

We first show that $\lim_s c_\tau(x, m)[s]$ exists for all $\tau \leq \beta$ and so all markers $\lambda(x, m, q) > \max_{\tau \leq \beta} \lim_s c_\tau(x, m)[s]$ may be enumerated into A . The value of $c_\tau(x, m)$ can be increased only when the mother node τ links to some worker σ with $\sigma < \beta'_{x,m}$ and subsequently defines more of the martingale m_τ . The priority tree is finite-branching, and so there are only finitely many nodes $\beta_{x,m}$. Consider

$$\langle e', k' \rangle = \max_{\beta_{x,m}} \{ \langle e, k \rangle \mid \sigma < \beta_{x,m} \text{ has mother } \tau \text{ and is devoted to } \mathcal{N}_{e,k} \}.$$

Let σ_0 be the node on TP devoted to $\mathcal{N}_{e',k'}$, and suppose that no node to the left of σ_0 is visited after stage s_0 . When we genuinely visit any node $\sigma \leq \sigma_0$ with mother node τ , the construction will check to see whether $m_\tau(\Delta_e^A[s] \upharpoonright d_\tau[s]) \geq k(\sigma)$. If so, we will play the d outcome. As $\sigma < \text{TP}$, the martingale will never decrease in capital and $c_\tau(x, m)$ cannot be increased again after travelling a link to some σ' devoted to $\mathcal{N}_{e,k(\sigma)}$. If $m_\tau(\Delta_e^A \upharpoonright d_\tau[s]) < k(\sigma)$,

then we will create a link from σ back to τ at stage $s > s_0$. If the link is permanent then $c_\tau(x, m)$ will never be increased. If we later define more of m_τ we will then increase $c_\tau(x, m)$. Again, as $\sigma < \text{TP}$, the martingale will never decrease in capital and $c_\tau(x, m)$ cannot be increased again after travelling a link to some σ' devoted to $N_{e,k(\sigma)}$. There are only finitely many worker nodes $\sigma \leq \sigma_0$ with mother node τ . Therefore $\lim_s c_\tau(x, m)[s]$ exists for all $\tau \leq \beta$.

We will restart all τ nodes with $e(\tau) > x$ below $\beta_{x,m'} \wedge \infty$. As $[m', \infty) \subseteq W_x$, the Π_2 test which measures whether $[m, \infty) \subseteq W_x$ will say “yes” infinitely many times for all $m \geq m'$. If a link (τ, σ) is placed over $\beta_{x,m'}$, and so $\beta_{x,m'} \wedge f \leq \sigma$, we will perform a scouting report when we arrive at τ . Suppose the link (τ, σ) is placed at stage s_0 and that s_1 is the least stage greater than s_0 at which the Π_2 test says “yes”. At the least τ -stage after s_1 , the scouting report will be successful, and $\beta_{x,m'} \wedge \infty$ will be accessible. We then will remove the link (τ, σ) and enumerate the marker $\lambda(x, m', t)$. In this way all of $\beta_{x,m'}$ ’s markers will eventually be enumerated.

Now suppose that $\beta_{x,m'} \wedge \infty \leq \tau < \beta_{x,m_1} < \sigma$. The node β_{x,m_1} has only the ∞ outcome, as it is below $\beta_{x,m'} \wedge \infty$ with $m' < m_1$. If β_{x,m_1} is on GTP then it will enumerate its markers whenever it is accessible and define Λ such that $\lim_t \Lambda^A(x, m_1, t) = 1$. If β_{x,m_1} is not on GTP then the link (τ, σ) must be permanent. As in the previous paragraph, $\beta_{x,m'} \wedge \infty$ will be accessible at infinitely many stages. When we arrive at τ and see the link, we will enumerate markers of the form $\lambda(x, m_1, q)$. Therefore all of β_{x,m_1} ’s markers will eventually be enumerated and $\lim_t \Lambda^A(x, m_1, t) = 1$. \square

This concludes the verification of the construction, and the proof of Theorem 1.8.

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